Cauchy - Frobenius lemma wrongly called Burnside

- its counting properties over orbits of groups acting on finite sets -

Talk about Marcel Wild Analytic Enumeration

of

isomorphic classes of linear binary matroids

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Bibliography

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Cauchy - Frobenius Counting Theorem also called Burnside Lemma

Outline of presentation

- ✓ **(Cauchy-Frobenius Lemma as Counting theorem**
- Transversal of orbits and the partition determined by a group action of a finite set.
- An Action of a group in a finite set equivalent to a permutation representation of the set.
- Fixed points, stabilizer groups , orbits
- Natural bijection between Orbits and Cosets of Stabilizers
- Standard Quotient Theorem:
- Lagrange Theorem
- Orbit-Stabilizer Theorem
- Proof Cauchy Frobenius Lemma
- (Frobenius-Cauchy- Polya) Burnside - Stabilizers and Fix points.
- Number of orbits equals the average of fix points.

(Frobenius-Cauchy- Polya)
Burnside Lemma

- ✓ tool that allows us to count the number of distinct items given a certain number of colors or other Characteristics.

It responds to questions like:

- ✓ How many distinct squares can be made with blue or yellow vertices?"

or

- ✓ How many necklaces with n beads can we create with clear and solid beads?"

- ✓ it will act as a picture function actually producing a polynomial that demonstrates what the different configurations are, and how many of each exist.

Cauchy-Frobenius Counting theorem also called Burnside Lemma

Burnside Cauchy Frobenius Lemma

Lemma: (the orbit-counting theorem)

result useful in taking account of symmetry when counting mathematical objects.

Let G be a finite group that acts on a set X.

$g \in G$, let X_g denote the set of elements in X that are fixed by g.

States that $|\frac{X}{G}| = \frac{1}{|G|} \sum_{(g) in G} |X_g|$

number of orbits (a natural number or $+\infty$) = average number of points fixed by g G.

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Transversal of orbits and the partition determined by a group action of a finite set.

Definition: The transversal of orbits

As G is an equivalence relation on X, a transversal F of the orbits yields a set partition of X, i.e, a complete dissection of X into the pairwise disjoint and nonempty subsets $G(t), t \in F$

$$X = \bigcup_{t \in F} G(t)$$

Hence the set of orbits will be denoted G

$$X := \{G(t) | t \in F\}$$

An Action of a group in a finite set equivalent to a permutation representation of the set.

Group Actions and Partitions

Each set partition of X gives rise to an action of a certain group on X.

let X_i ,where $i \in I$, an index set,

denote a partition of pairwise disjoint, nonempty sets which union is X.

An Action of G on a set X is equivalent to a permutation representation of G on X

it yields a set partition of X into orbits.

$$\bigoplus_i S_{x_i} := \{\pi \in S_x | \forall i \in I : \pi X_i = X_i\}$$

each set partition of X corresponds in a natural way to an action of

certain subgroup of the symmetric group S_x

which has blocks of the partition as its orbits.

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Relationship fixed points and Stabilizers
(Frobenius-Cauchy- Polya)
Burnside Lemma

Stabilizers and Fixed points

orbits

$G(x) \subset X$

X_g fix points

stabilizers.

$G_x \leq G$

Stabilizer of $x \in X$ is $G_x := \{g|gx = x\}$

$x \in X$ is fixed under Fixed point g in G iff $gx = x$.

The set of all fixed points of G is $X_g := \{x|gx = x\}$

The set of all fixed points of a subset S in G is $X_S := \{g \in S|gx = x\}$
If $S = G$ we call it Set of invariants.

we say x is a fixed point of g and
g fixes x.

stabilizer subgroup of x (also called the isotropy)
is the set of all elements in G that fix x:

Stabilizers of Elements in the same orbit

Definition: (Stabilizer of elements in the same orbit)

Let $x_1 . x_2 \in X$, and let $g \in G$ such that $x_1 = g.x_2$

Then G_{x_1} and G_{x_2} are related by $G_{x_1} = gG_{x_2}g^{-1}$.

$$g' \in G_{x_2} \text{ if and only if } g'.(g.x_2) = g.x_2$$
$$(g^{-1}g'g).x_2 = (g^{-1}g).x_2 = x_2;$$
$$g^{-1}g'g \in G_{x_2}$$

The stabilizers of elements in the same orbit are conjugate to each other.

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- ✓ **Natural bijection between Orbits and Cosets of Stabilizers**
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Natural bijection between Orbits and Cosets of Stabilizers

For a fixed x in X, consider map G to X

$$g \to g.x \text{ for all } g \in G.$$

image of this map is the orbit of x

the coimage is the set of all left cosets of G_x.

The standard quotient theorem of set theory

gives a natural bijection between G/G_x and Gx

given by $hG_x \to h.x$.

orbit-stabilizer theorem.

If G and X are finite then the orbit-stabilizer theorem, together with Lagrange's theorem, gives $|Gx| = [G : G_x] = |G|/|G_x|$.

This result can be employed for counting arguments.

The standard quotient bijection in between orbits and cosets of the Stabilizer

Standard Quotient Theorem:

The mapping $G(x) \to G/Gx : gx \to gG_x$ is a bijection ,

$gx = g'x \Leftrightarrow g^{-1}gx = g^{-1}g'x \Leftrightarrow x = g^{-1}g'x$

$\Leftrightarrow g^{-1}g' \in G_x \Leftrightarrow G_x = g^{-1}g'G_x \Leftrightarrow g'G_x = gG_x$

$$\Downarrow$$

Corollary: If G is a finite group acting on set X , then $x \in X$

$$|G(x)| = |G|/|G_x|$$

Corollary of the Standard quotient theorem on Centralizers and neutralizers.

Corollary:

If G is finite , g in G , and $U \leq G$, then

conjugacy classes of elements $|C^G(g)| = |G|/|C_G(g)|$

Centralizer

and

subgroups $|\tilde{U}| = |G|/|N_G(U)|$

Normalizer

The **centralizer** of a subset *S* of group (or semigroup) *G* is defined

$$\mathrm{C}_G(S) = \{g \in G \mid sg = gs \text{ for all } s \in S\}$$

The **normalizer** of *S* in the group (or semigroup) *G* is definec

$$\mathrm{N}_G(S) = \{g \in G \mid gS = Sg\}$$

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Lagrange Theorem:

Lagrange's Theorem *If G is a finite group and H is a subgroup of G, then $|H|$ divides $|G|$. number of distinct left cosets of H in G is $\frac{|G|}{|H|}$.*

⇩

$|G| = r|H|$. ⇩

$|a_iH| = |H|$ for each i,

⇩

$|G| = |a_1H| + |a_2H| + \cdots + |a_rH|$.

⇩ cosets are disjoint,

$G = a_1H \cup \cdots \cup a_rH$.

⇩

a in G, $aH = a_iH$ for some i $a \in aH$.

⇩

$a_1H, a_2H, \ldots, a_rH$

distinct left cosets of H in G.

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Orbit-Stabilizer Theorem

Corollary:

If G is a finite group acting on the set X , then for each $x \in X$ we have $|G(x)| = |G|/|G_x|$

⇑

$G(x)$ has the same number of elements as G / G_x

$|G(x)| = [G : G_x]$

⇑

$g * x \mapsto g G_x$

⇑

there is a well-defined bijection:

$G(x) \to G / G_x$

⇑

Standard Quotient Theorem

Proof Cauchy Frobenius Polya Lemma

The number of orbits of a finite group G acting on a finite set X is equal to the average number of fixed points:

$$|G\backslash\backslash X| = 1/|G| \sum_{ginG} |X_g|$$

$$|G| \sum_{t\in F}(1) = |G|.|G \backslash\backslash X|$$

number of orbits of finite group G acting on a finite set X

$$\sum_{x\in G(t)} |G(x)|^{-1} = |G(x)||G(x)|^{-1} = 1$$

$$GX := \{G(t)|t \in F\}$$

F is transversal

$$\sum_x |G||G(x)|^{-1} = |G| \sum_x |G(x)|^{-1} = |G| \sum_{t\in F} \sum_{x\in G(t)} |G(x)|^{-1}$$

Orbit-Stabilizer Theorem

Enumerating elements in the Stabilizer

$$\sum_x \sum_{g\in G_x} 1 = \sum_x |G_x| =$$

Enumerating fixed points in G x X

$$\sum_{g\in G} |X_g| = |\{(g,x) \in G \times X | g.x = x\}| = \sum_{g\in G} \sum_{x\in X_g} 1$$

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