# Brylawski – Lucas theorem of uniquely representable linear matroids

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# Bibliography


- Brilawski T, Lucas D, Uniquely representable combinatorial geometries, 1973
- Wild M., The Asymptotic number of inequivalent Binary codes and nonisomorphic binary matroids, May 1999
- Wild M., Consequences of the Brylawski-Lucas Th for binary matroids, 1996
- Wild M, Enumeration of binary and Ternary Matroids and other applications of the Brylawski-Lucas Theorem, Preprint, 1994

# Outline of the presentation

- ✓ Is matrix representation of Matroids Unique? (Introduction)
- Linear Space dependency and Matrices
- Row, Projective and Geometric equivalent matrix representations **.**
- Matroids Matrix and Bipartite graph representations**.**
- Basic incidence matrices and connectivity on Bipartite graphs
- Reduced B-basic matrix for matroid **G**
- Geometry equivalence subdeterminants criteria
- Matroid projective equivalent distinct matrix representations.
- Matroid row equivalent classes.

# Is matrix representation of matroids unique?

## Brylawski Lucas Theorem

Uniquely representable combinatorial geometries
T.Brylawski and D.Lucas problem

how a combinatorial abstract geometry ( Matroid), can be representable

in unique way, as a subset of a projective space

and what do these representations signify?

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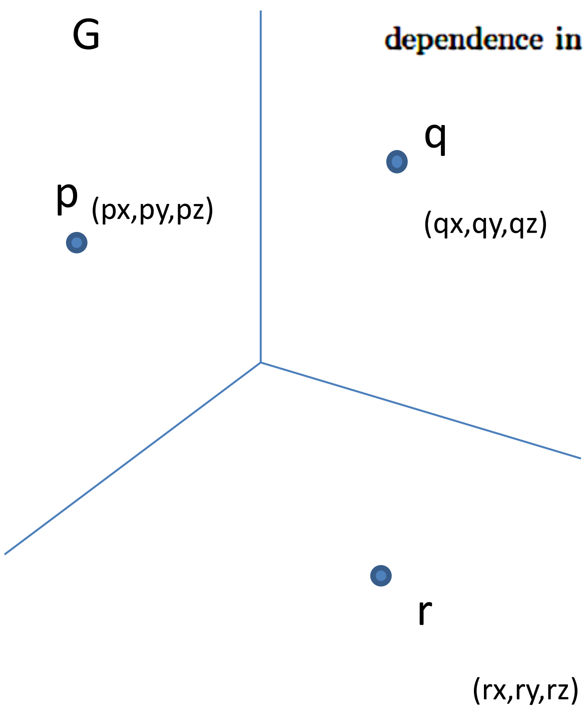
## Geometry dependency in the space and linear dependency in matrices

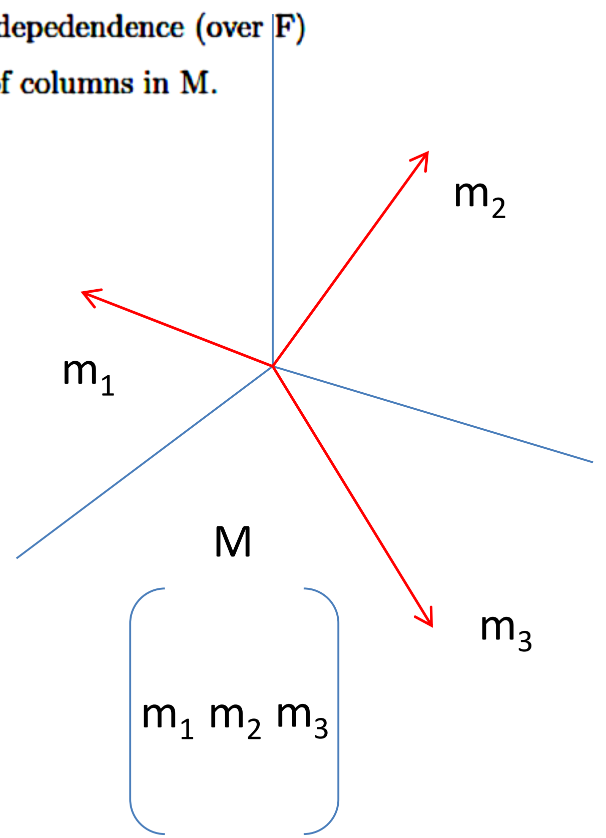
**Definition: (representable Pregeometry)**

A combinatorial geometry (Matroid) G ⟷ representable over a field F

points of G ⟷ (one to one) n column vectors of a matrix M with entries in F

dependence in G ⟷ linear depedendence (over F) of columns in M.



## Row, Projective and Geometric equivalence among matrix representations

$M \overset{R}{\sim} M'$:

- Add scalar multiple of a row to another
- Permuting 2 rows
- Multiply a row by a scalar ($\neq 0$)

**Definition: (Projective operation)**

$M \overset{P}{\sim} M'$:

- elementary row operation,
- multiplication of a column by nonzero scalar,
- a removal of a zero row.

**Definition: (Geometrically equivalent Matrices)**

**projective operation does not affect column dependence,**

**matrices**

**M** ⟷ **M'**

**differ by a series of projective operations**

$M \overset{P}{\sim} M'$

⇓

**represent same pregeometry ( Matroid).**

$M \overset{G}{\sim} M'$

$M \overset{R}{\sim} M' \rightarrow M \overset{P}{\sim} M \rightarrow M \overset{G}{\sim} M'$

# Projective equivalence in between matrix representations

$$M \overset{P}{\sim} M'$$

**Definition: ( Projectively equivalent matrices)**

**Two matrices**

**projectively equivalent if**

$$M' = NMD,$$

**N** — **nonsingular $r \times r$ matrix**

**D** — **nonsingular $n \times n$ diagonal matrix.**

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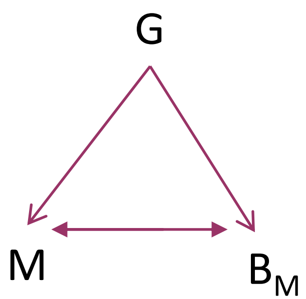
# Coordinatization among Matrix representation and Bipartite graph representation




$M = (I A)$

coordinatization (representation) when $a_{ij} \neq 0$.

G

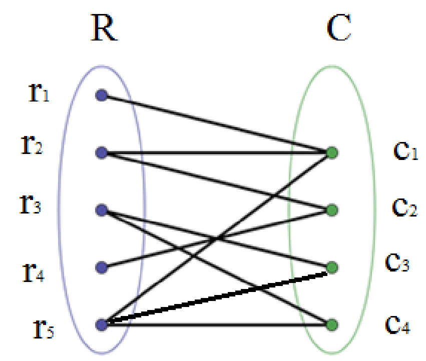
$B_M$

bipartite graph

adjacency matrix $\bar{A}$ where $\bar{a}_{ij} = 1$

$$
\begin{array}{c|cccc}
 & c_1 & c_2 & c_3 & c_4 \\
\hline
r_1 & 1 & 0 & 0 & \\
r_2 & 1 & 1 & 0 & \\
r_3 & 0 & 0 & 1 & \\
r_4 & 0 & 1 & 0 & \\
r_5 & 1 & 0 & 1 &
\end{array}
$$

A




B

Definition: ( Coordinatizing path of M or A. )

A basis of $B_M$

Definition: ( Coordinatizing circuit of M or A. )

A basic circuit of $B_M$

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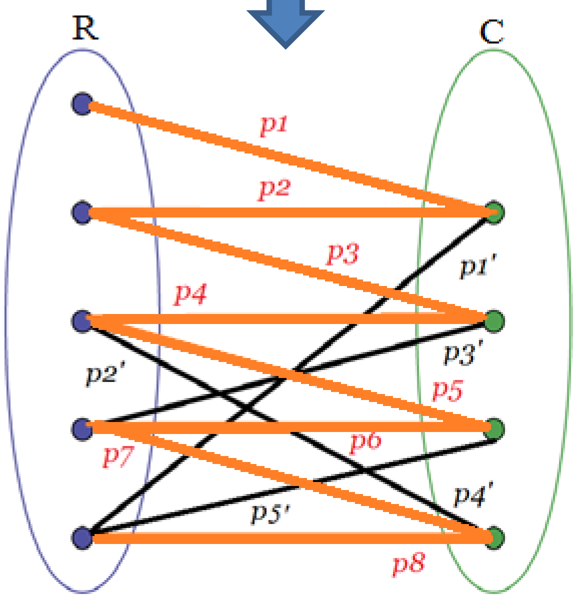
# C basic incidence matrix representations.

**C**

$$
\begin{array}{c|cccc}
 & p'_1 & p'_2 & \dots & p'_{n-r} \in G - B \\
\hline
 & 0 & 0 & 0 & 0 \\
p_1 & 1 & 0 & 0 & 0 \\
p_2 & 1 & 0 & 0 & 0 \\
\cdot & 1 & 0 & 1 & 0 \\
\cdot & 1 & 1 & 1 & 0 \\
\cdot & 1 & 1 & 1 & 1 \\
p_r & 1 & 1 & 0 & 1 \\
\in B & 1 & 1 & 0 & 1
\end{array}
$$




Definition: (C basic incidence matrix)

Given

a Pregeometry (Matroid) G,

a basis B

a path P,

zero-one matrix

rows indexed by

: $(p_1, p_2, \dots, p_r)$

columns indexed by points

$(p'_1, p'_2, \dots, p'_{n-r})$ in $G - B$,

with

entries $c_{ij}$

s.t.

$c_{ij} = 1$

if $p_i$ is

unique circuit

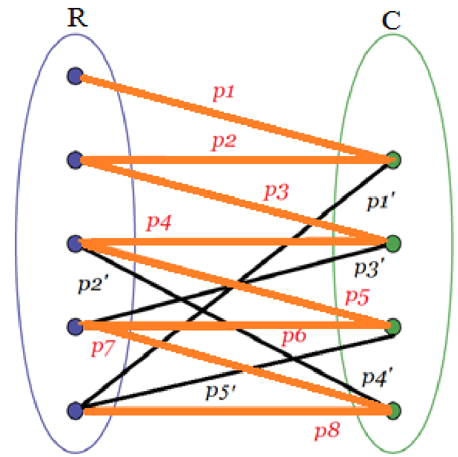
(minimal dependent set)

$p_i$ in $B \cup p_j$

## B basic incidence matrix representation and connectness

**C**

$$
\begin{array}{c|cccc|c}
 & p'_1 & p'_2 & \cdots & p'_{n-r} & \in G - B \\
\hline
 & 0 & 0 & 0 & 0 & \\
p_1 & 1 & 0 & 0 & 0 & \\
p_2 & 1 & 0 & 0 & 0 & \\
\vdots & 1 & 0 & 1 & 0 & \\
 & 1 & 1 & 1 & 0 & \\
 & 1 & 1 & 1 & 1 & \\
p_r & 1 & 1 & 0 & 1 & \\
\in B & 1 & 1 & 0 & 1 &
\end{array}
$$




geometry(Matroid) G,

If C is B-basic circuit incidence matrix of G,

then the number of connected components

of G equals $r' + c' + k$,

r' is the number of zero rows of C,

c' is the number of zero columns of C,

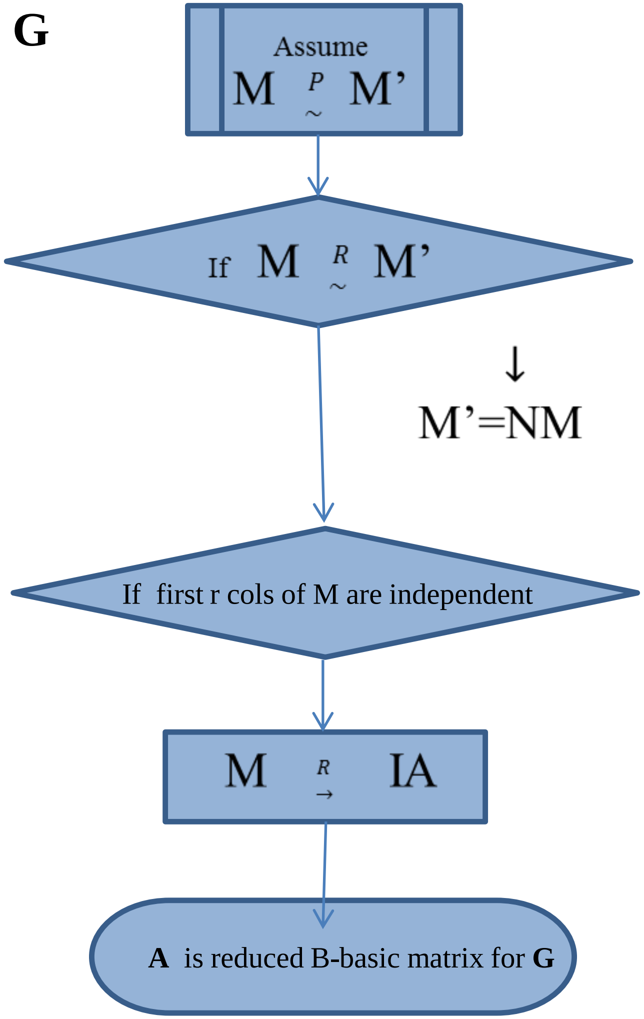
k is the number of blocks of C after zero rows and

and

columns have been eliminated.

**How to get a reduced b-basic matrix for a matroid**

Reduced B-basic matrix for **G**



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# Geometric equivalence and subdeterminants of adjacent and incidence matrices.

Proposition:

If $M = IA$ and $M' = IA'$

$$M \underset{\sim}{G} M'$$

iff

every subdeterminant of A $D \equiv 0$

exactly when the

subdeterminant of A' $D' \equiv 0$

entry $a_{ij} = 0 = a'_{ij}$

iff

$c_{ij} = 0$

circuit incidence matrix

**C**

of G

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{13} & D \end{vmatrix}$$

$$|A'| = \begin{vmatrix} a'_{11} & a'12 \\ a'13 & D' \end{vmatrix}$$

$$\begin{array}{c|cccc} & p'_1 & p'_2 & p'_3 & p'_4 \\ \hline p_1 & & & & \\ p_2 & & & & \\ p_3 & & & & \\ p_4 & & & 0 & 0 \\ p_5 & & & 0 & 0 \end{array}$$

C

## Number of distinct matrix representations under Projective equivalence

Proposition:

Let G be a matroid (pregeometry)

rank r cardinality n,

k connected components.

P coordinatizing path of A

If $M = IA$

represents G over F

$if \mid F \mid = q,$

$\exists$

$(q - I)^{n-k}$ distinct matrices

$[IA'] = M' \overset{P}{\sim} M$

$\exists$

$(q^r - I)(q^r - q)\ldots(q^r - q^{r-1})(q - I)^{n-k}$

distinct $r \times n$ matrices,

$PA' = M' \overset{P}{\sim} M$

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# Number of matrices and equivalence classes under row equivalence

Corollary:

If G representable over F, $|F| = q$,

each projective equivalence class

with

$PA'$ $r \times n$ representations

is partioned into

$(q - I)^{n-k}$ distinct row equivalence classes

each class contains $\prod_{i=0,r-1} q^r - q^i$

$r \times r$ non singular distinct matrices over $F_q$.