Forbidden minors of P-representability from F. Matus formalism

By Alexander Erick Trofimoff
Graduate Research Assistant
EE PhD Candidate Drexel U.
ECE dept Fall 2016

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Algebra of Minors on Probabilistically representable Matroids

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Algebra of Minors on Probabilistically representable Matroids

Applying the algebra of minors we can,
since minors of probabilistically
representable matroids are of the same type,
translate theorems for binary, ternary and regular matroids
into the probabilistic language

Minors of Localizable
CI relations
(based on planar Graphs Kuratowski Th
and Tutte's characterization of Graphic Matroids)

Given subsets $L, M \subset N$,
The minor $N_{\mathcal{M}}$ is the ternary relation on L
Consisting of all triples (I, J, K) s.t.
$I \cup J \cup K \subset L$
And
$(I, J, K \cup M) \in \mathcal{M}$
If M=∅ $N_{\mathcal{M}}$ is called Restriction
If M+N=L $N_{\mathcal{M}}$ is called a Contraction of $\mathcal{M}$

Minors of Localizable CI relations (based on planar Graphs Kuratowski Th and Tutte's characterization of Graphic Matroids) Examples:

Having a CI $\mathcal{M}$={ ({1},{2},{4}) , ({2},{3},{4}) , ({1},{3},{∅}) }

On $N = \{1,2,3,4\}$

Given subsets $L, M \subset N$,

The relation $N'_{\mathcal{M}}$={({1},{3},{0})} on L ={1,2,3}

Is a restriction of $\mathcal{M}$ (M=∅)

The relation $N''_{\mathcal{M}}$={ ({1},{2},{0}) , ({2},{3},{0}) }on L ={1,2,3}

Is a contraction of $\mathcal{M}$ (M={4})

The relation $N'''_{\mathcal{M}}$={ ({1},{2},{0}) } on L ={1,2}

Is a contraction of $\mathcal{M}$ (M={4}) neither a restriction

Nor a contraction of $\mathcal{M}$

Minors of p-representable matroids .

Minors via Restrictions.

Let $M \subset \mathcal{J}(N)$ be a global ternary relation and $L \subset N$.

$\mathcal{J}(N):$ all $(IJ|K), s.t.\ I, J, K \subset N,\ I \cap J \cap K = \emptyset$

The *restriction* of M to L

is given by

$\mathrm{re}_L\, M = M \cap \mathcal{J}(L) = \{(I, J|K) \in M : I \cup J \cup K \subset L\}$

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Minors of p-representable matroids .

Minors via Contractions.

contraction to L is

$\text{co}_L\mathcal{M} = \{(I,J|K) \in \mathcal{I}(L);\ (I,J|K(N - L)) \in \mathcal{M}\}$

$(I,J|K(N - L)) = (I,J|[K \cup (N - L)])$

if L and M are disjoint subsets of N,

the relation $\text{co}_L\ \text{re}_{LM}\mathcal{M}$ on L

is called a *minor* of $\mathcal{M}$.

The above minor can be equivalently given by

$\text{re}_L\ \text{co}_{N - M}\mathcal{M} = \{(I,J|K) \in \mathcal{I}(L); (I,J|KM) \in \mathcal{M}\}$

$(I,J|KM) = (I,J|[K \cup M])$

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Minors of p-representable matroids .

Minors via Sequences of Restrictions/Contractions.

whence minors of a ternary relation $\mathcal{M}$ on N are
ternary relations on subsets of N
constructed from $\mathcal{M}$ by any
sequence of restrictions and contractions.
It can be frequently
All minors can be constructed by performing restrictions and contractions
by one-element sets.
We speak of *proper* minors if $L \neq N$ and of n-minors,
n-graphoids, etc. if they are relations on a set of n elements

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Set of Loops of a rank function.

An element $i \in N$
is called a loop of a rank function
r when $r(i) = 0$.
That is,
$i \in N$ is a loop of a Matroid $\mathcal{L}$ if $(i|\emptyset) \in \mathcal{L}$
Then
The loopless relation
$\mathcal{L} \subset \mathcal{J}(L)$ is a matroid
Iff
$r_{\mathcal{L}}$ =Max{$|J|$, $J \subset I$, $\mathcal{R}$ $(J) \cap \mathcal{L} = \emptyset$} $I \subset N$
Is semimodular
and
$\mathcal{L}$=[$r_{\mathcal{L}}$]

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Set of Loops of a rank function.

For a matroid $\mathcal{L}$ with a set of loops $\lambda(\mathcal{L})$

$r_{\mathcal{L}}(I)= r^{\mathcal{L}}(I)+|I \cap \lambda(\mathcal{L})|, I \subset N$

$r^{\mathcal{L}}$ is the rank function of $\mathcal{L}$

$r_{\mathcal{L}}$ is the rank function of a matroid obtained from $\mathcal{L}$

By the conversion of all its

Loops into isthmuses.

$i \in N$ is an isthmus or a coloop

If $r(i) = 1, and\ r(N) = r(i) + r(N - i)$.

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Restrictions and Contractions of Minors.

LEMMA

Minors of p-representable semimatroids are p-representable.
Minors of strongly p-representable matroids are strongly p-representable with larger p-characteristic sets.

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Minors of p-representable matroids.

LEMMA

Proof:

Let ξ =(ξ_i)$_{i \in N}$ be the p-representation of a semimatroid
$\mathcal{L}$, as re$_I$($\mathcal{L}$) , $I \subset N$, and co$_I$($\mathcal{L}$), $N - I \subset \lambda(N)$
are representable by $\xi_{I,}$
We only need to show that
Contractions of $\mathcal{L}$ by nonloops are p-representable
For $i \in N - \lambda(\mathcal{L}), k \in N - i,$
Construct $\forall$ xi value of ξ_i, new r.v.

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Minors of p-representable matroids.

LEMMA

Proof:

$\xi_k^{(x)}$=$(\xi_k^{(x_i)})_{k \in K}$, by restricting ξ_k, to event ξ_i=x_i ,

,i.e.,

Pr($\xi_k^{(x_i)}$= y_k)=$\frac{(x_i y_k)}{(x_i)}$, $y_k \in X_k$,

all $\xi_k^{(x_i)}$, $x_i \in X_i$, are mutually independent.

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LEMMA

Minors of p-representable matroids.

Proof:

$\mathrm{co}_k(\mathcal{L})$, of semimatroid $\mathcal{L}=[h_\xi]$
is determined by

$$g(J) = h_{\xi(ij)} - h_{\xi(i)}$$

$$= -\sum_{x_{iJ} \in X iJ} (x_{iJ}) \log(x_{iJ}) + \sum_{x_i \in Xi} (x_i) \log(x_i)$$

$$= \sum_{x_{iJ} \in X iJ} (x_{iJ}) \log\left(\frac{x_{iJ}}{x_i}\right), J \subset K,$$

$g(J)$: conditional entropy of ξ_j given ξ_i

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Restrictions and Contractions of Minors.

LEMMA

Proof:

$$g(J) = -\sum_{x_i \in Xi}(x_i)\sum_{y_J \in Xj}\frac{(x_i y_j)}{(x_i)}\log\frac{(x_i y_j)}{(x_i)} + \sum_{x_i \in Xi}(x_i) h_{\xi_k}{}^{(xi)}(J),$$

$J \subset K$,

$g(J)$ is a convex combination of

funtions $h_{\xi_k}{}^{(xi)}$, $x_i \in X_i$,

This implies

$\text{co}_k(\mathcal{L}) = [g] = \bigcap_{x_i \in X_i}[\ h_{\xi_k}{}^{(xi)}]$

And Thus,

$\eta = (\eta_k)_{k \in K}, \eta_k = (\xi_k{}^{(x_i)})_{x_i \in X_i}$

Is a p-representation of $\text{co}_k(\mathcal{L})$.

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Restrictions and Contractions of Minors.

LEMMA

Proof:

If ξ is a strong p -representation

Of matroid $\mathcal{L}$ of degree u

Then all the systems

${\xi_k}^{(x_i)}$ has the same entropy function

Matching $with\ g$

They all are strong p-representation of matroid $\text{co}_k(\mathcal{L})$.

With degrees u

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Matroids and Semimatroids Duality and their Minors.

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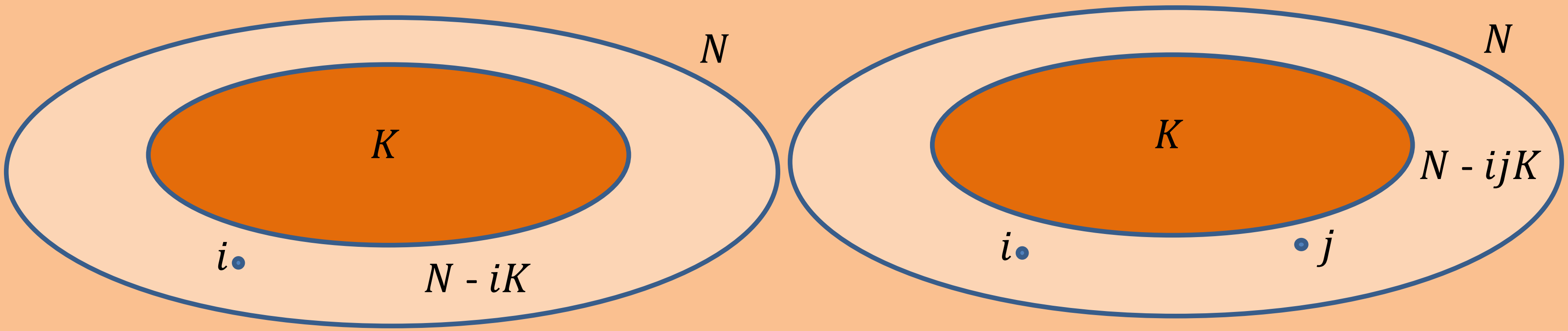
Matroid Duality and Minors.

To any matroid $\mathcal{L} \subset \mathfrak{S}(N)$ we denote

$$\mathcal{L}^{\top} = \{(i \mid K) \in \mathcal{Q}(N) ; (i \mid N - iK) \notin \mathcal{L}\} \cup \{(ij \mid K) \in \mathcal{R}(N); (ij \mid N - ijK) \in \mathcal{L}\}$$

if r is the rank function of $\mathcal{L}$ then $\mathcal{L}^{\top}$ is the matroid with the rank function given by $r^{\top}(I) = |I| + r(N - I) - r(N),\ I \subset N.$

$$\mathcal{L} = \mathcal{L}^{\top\top} \quad \forall \mathcal{L} \subset \mathfrak{S}(N)$$



$$\mathcal{L}^{\top} = \{i \mid K : (i \mid N - i \cup K) \notin \mathcal{L}\} \cup \{ij \mid K : ij \mid N - (i \cup j \cup K) \in \mathcal{L}\}$$

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Semi-Matroid Duality .

if $\mathcal{L} = [h]$, $h \in H(N)$, is a semimatroid

$[h] : \{(ij \mid K) \in \mathfrak{S}(N);\ \Delta h(i, j \mid K) = 0\}$

then $\mathcal{L}^{\intercal} \cap \mathcal{R}(N)$;
is also a semimatroid, e.g.
it descends from the function

$h^{\intercal} + \sum_{i\in N} g_i$ where

$h^{\intercal}(I) = \sum_{i\in I} h(i) + h(N - I) - h(N),\ I \subset N.$

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Content

Restrictions and Contractions
in
Semimatroids

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Minors of localizable relations

Minors are defined through restrictions and contractions and discussed the classes of semigraphoids, pseudographoids and graphoids from this point of view.

Restrictions and Contractions and Minors.

The restrictions and contractions of semimatroids
are semimatroids.
A relation $\mathcal{K}$ on $I \subset N$ is called **a minor** of a relation $\mathcal{L}$ on N if
∃ a set J s.t. $I \subset J \subset N$ and $\mathcal{K}$ = $\mathrm{re}_I(\mathrm{co}_J(\mathcal{L}))$.
The order of the operations is not substantial and thus
minors of minors are minors.

The Duality in between restrictions and contractions.

If $\mathcal{L}$ is a relation on N and $I \subset N$ then we shall denote by
$\mathrm{Re}_I\ (\mathcal{L}) = \mathcal{L} \cap \mathfrak{S}\ (I)$ its restriction to I
and
by $\mathrm{co}_I(\mathcal{L}) = \{(\ ij \mid K) \in \mathfrak{S}(I);\ (ij \mid K(N - I)) \in \mathcal{L}\}$ its contraction by $N - I$ (both considered for relations on I).
Immediately, for $I \subset J \subset N$ we establish $re_I = re_I\, re_J$, $co_I = co_J\, co_I$
and $re_I\, co_J = co_I\, re_I\, r_{I(N-J)}$.
These two operations are dual in the above sense
$(\mathrm{re}_I(\mathcal{L}^{\top}))^{\top} = \mathrm{co}_I(\mathcal{L})$,
$I \subset N$, :: $\mathcal{L} \subset \mathfrak{S}(N)$.

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Restrictions and Contractions of Matroids.

The **restriction** to I of a matroid with the rank function r
is the (sub)matroid with rank function $r|_{\mathcal{P}(I)}$
and its **contraction** by $N - I$ is the matroid on I
with the rank function
having the values $r(J(N-I)) - r(N-I), J \subset I$.
The restrictions and contractions of semimatroids
are semimatroids.

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Content

Minors Characterization of Classes of CI-relations in Ternary relations .

CI-relations as Ternary relations .

CI-relations are ternary relations $\mathcal{M}$ consisting of triples (I, J, K) of subsets I, J and K of a finite set N.
CI-relations are studied here from mathematical logic influenced by the integrity constraint theory of databases.

CI-relations as Graphs and Matroids .

CI-relations are, however, algebraical structures rich enough to contain graphs and matroids.
we are going to study localizable CI-relations through the notion of *minor* familiar from the graph and matroid theories.

Why use Minors to Analyze CI-relations? .

Concept: Minors of CI-relations are their natural subconfigurations.
Goal: recognize the relations of the class only through their, possibly small, subconfigurations that are characteristic for and inherent to the class, or equivalently, through their small defects or obstructions – forbidden subconfigurations of the class.
Use:
Minors should henceforth provide a complementary tool to the axiomatic approach.

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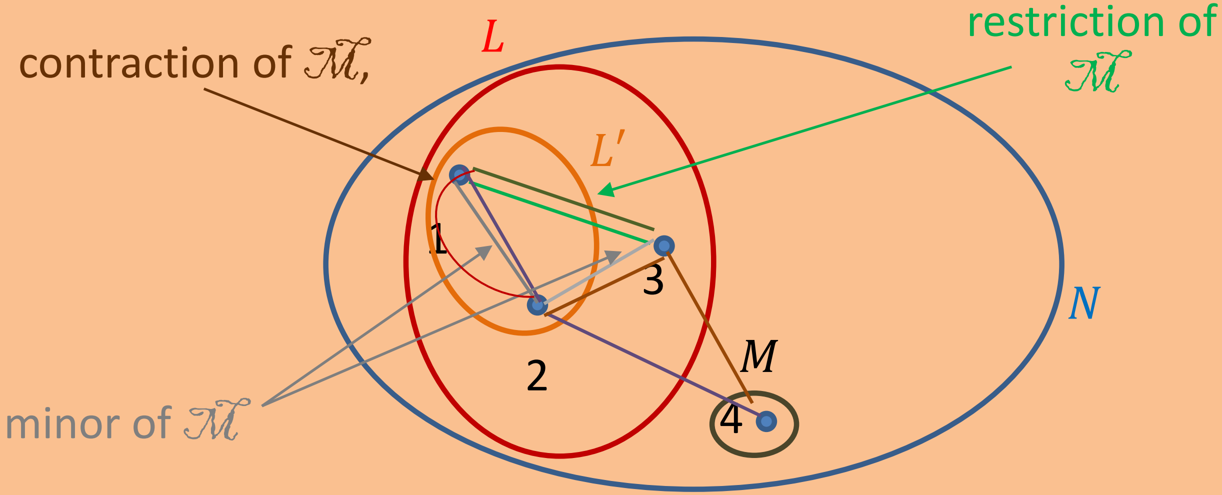
Minor defined by Matroid Theory.

Given disjoint subsets L and M of N, the corresponding minor of a ternary relation $\mathcal{M}$ on N is the ternary relation on L consisting of all triples (I, J, K) such that $I \cup J \cup K \subset L$ and $(I, J, K \cup M) \in \mathcal{M}$.
If $M = \emptyset$ the minor is called a restriction of $\mathcal{M}$ and if $M + L = N$ a contraction of $\mathcal{M}$.

Minors of CI-relations.

Ex: having the relation $\mathscr{M}$ = {({1}, {2}, {4}), ({2}, {3}, {4}), ({1}, {3}, {∅})}
on N = {1, 2, 3, 4},
the relation {({1}, {3}, {∅})} on L = {1, 2, 3} is a minor of $\mathscr{M}$ (even a restriction of $\mathscr{M}$, w.r.t M = ∅), the relation {({1}, {2}, {∅}), ({2}, {3}, {∅})} on L = {1, 2, 3} is a minor of $\mathscr{M}$ (even a contraction of $\mathscr{M}$, w.r.t M = {4}) and the relation {({1}, {2}, {∅})} on L' = {1, 2} is a minor of $\mathscr{M}$, w.r.t M = {4}, that is neither a restriction nor contraction of $\mathscr{M}$.



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Hereditary Global CI relations.

Localizible Global CI relation.

Def.

Let $\mathcal{M} \subset \mathcal{I}(N)$,

We are interested only in the relations $\mathcal{M}$ s.t.

$(I\,,J|\;K \in \mathcal{M}) \Leftrightarrow \{(\forall\; j \in J)\;(\forall\; i \in I)(\forall\; L \supset K)\; L \subset IJK\text{-}ij \Longrightarrow i\,,j|\;L \in \mathcal{M})\}$

That we call localizable.

$\mathcal{I}(N)$ are all the $(IJ\ |K)$, where $I, J, K\ \subset N$,

$s.t.\, I \cap J \cap K = \emptyset$

Localizing Operator
And
Global CI relations
That are reconstructible.

Let $\mathcal{M} \subset \mathcal{I}(N)$ and $\mathcal{L} \subset \mathcal{R}(N)$

We can reconstruct any localizable global relation $\mathcal{M}$ from

$\mathrm{Loc}(\mathcal{M}) = \mathcal{M} \cap \mathcal{R}(N) = \mathcal{L}$,

Using the formula

$$\mathcal{M} = gl\,\mathcal{L} = \{I, J \mid K \subset \mathcal{I}(N); (I, J \mid K)_* \subset \mathcal{L}$$

$$(I, J \mid K)_* = \{i, j \mid L \in \mathcal{R}(N),\; j \in J, i \in I; K \subset L \subset IJK\text{-}ij\}$$

Global CI relations that are Localizible.

The global relation $\mathcal{M}$ is localizable

iff

$$gl(\text{loc}\ (\mathcal{M})) = \mathcal{M}$$

$$(\forall i \in I - K)(\forall\ j \in J - K)(\forall\ L\ \subset N)\ (K \subset L \subset IJK\text{-}ij \Longrightarrow i\ , j|\ L \subset \mathcal{L})\}$$

Hereditarly Global CI relations .

The global relation $\mathcal{M}$ is localizable
iff gl(loc ($\mathcal{M}$))= $\mathcal{M}$
Every $\mathcal{M}$ contains the triplets $I\,,J|\;L$ with empty I or J
Satisfying
$I\,,JK|\;L \in \mathcal{M} \Longrightarrow$
$[(I, J\,|KL) \in \mathcal{M}$ and $(I, K\,|L) \in \mathcal{M}]$
These global relations are called Hereditary.

Semigraphoids.

Are Hereditary global relations that satisfy also
The converse implication
$$[(I, J \mid KL) \in \mathscr{M} \text{ and } (I, K \mid L) \in \mathscr{M}]$$
$$\Longrightarrow$$
$$(I, JK \mid L) \in \mathscr{M}$$
That is known as the Semigraphoid Axiom.

Pseudographoids.

Are Hereditary global relations that satisfy also
The converse implication

$$[(I, J \mid KL) \in \mathscr{M} \text{ and } (I, K \mid JL) \in \mathscr{M}]$$
$$\Longrightarrow$$
$$(I, JK \mid L) \in \mathscr{M}$$

That is known as the Pseudographoid Axiom.

Graphoids.

Are Hereditary global relations that are Semigraphoids and
At the same time Pseudographoids

Lemma.

Semigraphoids, pseudographoids and graphoids are localizable

Content

Minors obtained via Restrictions-Contractions of CI-relations.

Geometric Representation of CI relations.
(Rules)

1) Circles are connected by a kind of line, labeled by lists of subsets.
2) Dotted lines: If i, j triples $(i,j|K) \in \mathcal{L}$ whenever $K \subset N - ij$.
3) Full lines: If $\forall\ K \subset N - ij$ triple $(i,j|K) \notin \mathcal{L}$
4) If No, decide freely the kind of line to use.
5) Dotted line from i to j $\exists$ list of K s.t. $(i,j|K) \in \mathcal{L}$ attached to it.
6) Full line, means the complement of this list must be provided.
7) The symbol $*$ abbreviates set $N - ij$ on line between i and j.
8) Dotted and full lines match ‘conditional independences and dependences’ of $\mathcal{L}$.
9) Trivially $\mathcal{L} = \emptyset$ is a set of empty circles every 2 connected by a full line,
10) $\mathcal{L} = \mathcal{R}(N)$ ($\mathcal{M} = \mathcal{Q}(N)$) similarly, replacing all full lines by the dotted ones.

Example.

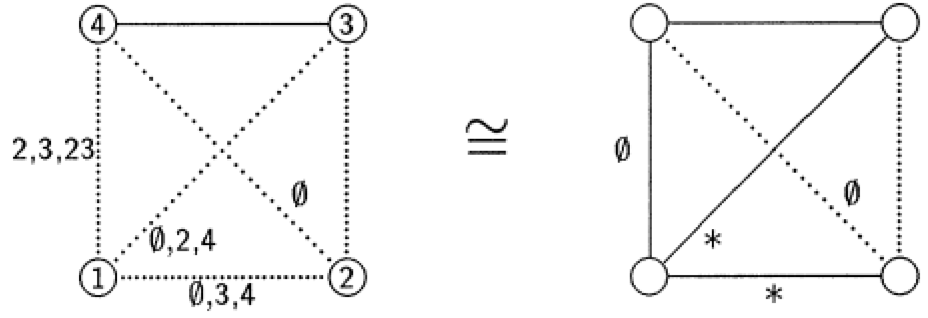

Figure 1. An example of visualization of a local(izable) relation.

Let N = {1, 2, 3, 4} and assume that a local relation $\mathcal{L}$ consists of the triples (1, 2|∅), (1, 2|3), (1, 2|4), (1, 3|∅), (1, 3|2), (1, 3|4), (1, 4|2), (1, 4|3), (1, 4|23), (2, 3|∅), (2, 3|1), (2, 3|4), (2, 3|14), (2, 4|∅).

Restrictions and Contractions of CI-relations.

Let $\mathcal{M} \subset \mathcal{J}(N)$ be a global ternary relation and $L \subset N$.
The restriction of $\mathcal{M}$ to L is given by
$\mathrm{re}_L\, \mathcal{M} = \mathcal{M} \cap \mathcal{J}(L) = \{\, (I,J|K) \in \mathcal{M};\ I \cup J \cup K \subset L\}$
and its contraction to L is
$\mathrm{co}_L\, \mathcal{M} = \{\, (I,J|K) \in \mathcal{J}(L);\ (I,J|K(N-L)) \in \mathcal{M}\}$.
Where L and M are disjoint subsets of N, the relation
$\mathrm{co}_L \mathrm{re}_{LM}\, \mathcal{M}$ on L is called a minor of M.
One observes immediately that the above minor can be equivalently
given by $\mathrm{re}_L\, \mathrm{co}_{N-M}\, \mathcal{M} = \{\, (I,J|K) \in \mathcal{J}(L);\ (I,J|KM) \in \mathcal{M}\}$,

$\mathcal{J}(N)$ are all the $(IJ\ |K)$, where $I, J, K \subset N$,
$s.t.\ I \cap J \cap K = \emptyset$

Minors are Obtained from Contractions And Restrictions.

Minors of a ternary relation $\mathcal{M}$ on N are ternary relations
on subsets of N constructed from $\mathcal{M}$ by
any sequence of restrictions and contractions.
Ex: ‘contraction (restriction) by $N - L$’ .
All minors can be constructed by performing restrictions and contractions
by one-element sets.
We speak of proper minors if $L \neq N$ and of n-minors, n-graphoids, etc.
if they are relations on a set of n elements.

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Forbidden minors of P-representability

Minors Representation.

Example.

Ex: Let $\mathcal{L}$ be the local 4-relation from Previous Example.
Diagrams of two minors of $\mathcal{L}$ are visualized on figure 2

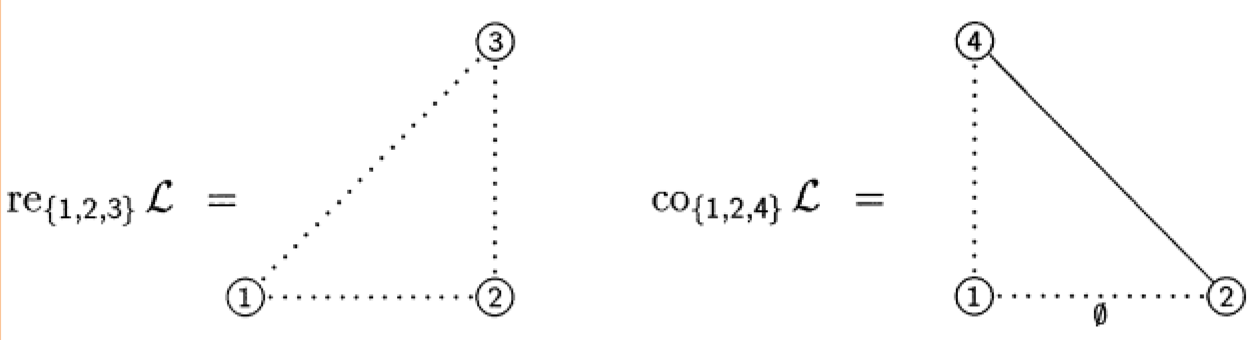

Figure 2. An example of restriction and contraction.

Hereditary in Minors of CI-relations.

Lemma.

If a relation $\mathscr{M}$ is hereditary (localizable) then all its minors are hereditary (localizable). Minors of semigraphoids, pseudographoids and graphoids are semigraphoids, pseudographoids and graphoids, respectively.

Minor Closed Classes and Forbidden Minors of Classes.

All minors of CI-relations from
(hereditary, localizable, semigraphoid, graphoid, pseudographoid)
classes remain in the same class.
Such classes are called **minor-closed**.
If a CI-relation does not belong to a minor-closed class
but all its proper minors are in the class then
It is called a **forbidden minor of the class**.
Any minor-closed class can be uniquely described by
(the isomorphism classes of all) its forbidden minors.
A relation belongs to such a class iff
no its minor is isomorphic to a forbidden minor of the class.

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Compulsory Minors of Classes.

If a minor-closed class of CI-relations has a forbidden n-minor for a fixed number n then the n-relations of this class are called the compulsory n-minors.
For different values of n compulsory or forbidden minors will be examined according to what is more economical.

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Minor characterization of graph CI-relations

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Proposition.

A ternary localizable relation $M \subset \mathcal{Q}(N)$ is a graphoid (semigraphoid, pseudographoid)
iff
all its 3-minors are
graphoids (semigraphoids, pseudographoids, respectively).

Ex.

The hereditary relation $gl((12, 34|\emptyset)_*) - \{(12, 34|\emptyset)\}$ on N = {1, 2, 3, 4}
is not localizable and all its 3-minors are graphoids.
The assumption of localizability in Proposition 1 is therefore substantial

Forbidden minors of P-representability

Consequence.

A localizable relation $M \subset \mathcal{T}(N)$ is a graphoid iff each nontrivial 3-minor of M is in one of the six isomorphism classes of figure 3.

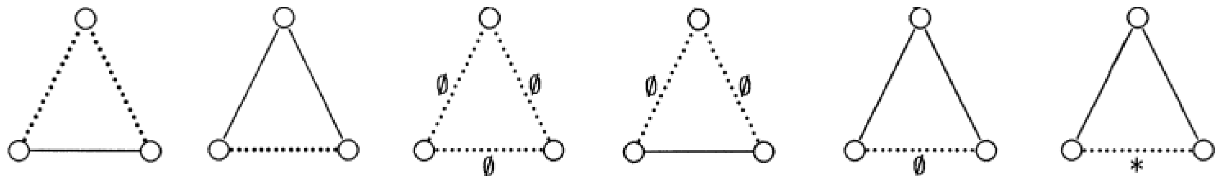

Figure 3. Nontrivial 3-graphoids: nontrivial compulsory 3-minors of graphoids.

Consequence.

A localizable relation $M \subset \mathcal{T}(N)$ is a semigraphoid iff each its nontrivial 3-minor is isomorphic to a 3-semigraphoid from figure 3 or from figure 4.

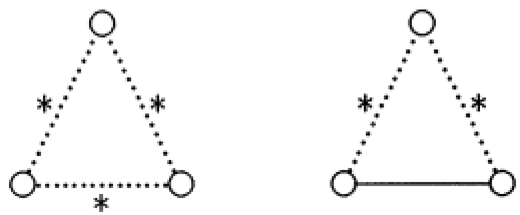

Figure 4. 3-semigraphoids that are not graphoids.

Consequence.

A localizable relation $M \subset \mathcal{Q}(N)$ is a pseudographoid iff it has no 3-minor isomorphic to the relations from figures 4 and 5.

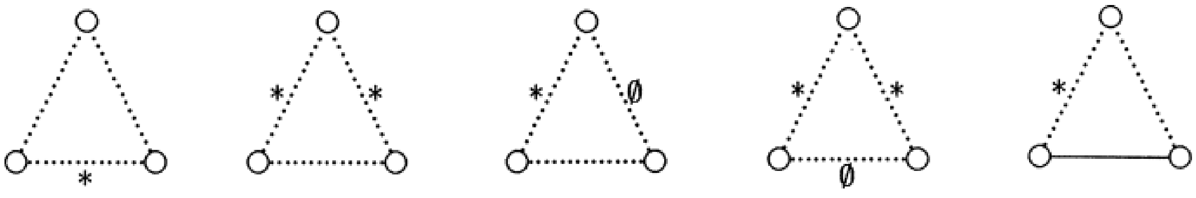

Figure 5. Localizable forbidden 3-minors of pseudographoids.

Graphoids and Semigraphods on local CI-relations

$\mathcal{L} \subset \mathcal{R}(N)$ is a semigraphoid if it satisfies

$[(i, j|kL) \in \mathcal{L}$ and $(i, k|L) \in \mathcal{L}] \Longrightarrow [(i, k|jL) \in \mathcal{L}$ and $(i, j|L) \in \mathcal{L}]$

$\mathcal{L} \subset \mathcal{R}(N$ is a pseudographoid if it satisfies

$[(i, j|kL) \in \mathcal{L}$ and $(i, k|jL) \in \mathcal{L})] \Longrightarrow [(i, j|L) \in \mathcal{L}$ and $(i, k|L) \in \mathcal{L}$]

$\mathcal{L} \subset \mathcal{R}(N$ is a Graphoid if it obeys both implications as before.

Content

Forbidden minors of P-representability

Separation Graphoids local CI-relations

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Separation Graphoids local CI-relations

Chordal graph: all cycles of four or more vertices have a *chord*, which is an edge that is not part of the cycle but connects two vertices of the cycle.

Tree: undirected graph in which any 2 vertices are connected by *exactly one* path. Any acyclic connected graph is a tree. A **forest** is a disjoint union of trees.

Cutsets and

Separation Graphoids of Undirected simple Graphs

Let G be an undirected simple (no loops and multiple edges) graph vertex set N and let $\langle G\rangle = \{(i, j|K) \in \mathcal{R}(N);\ K$ is a cutset of i and j in $G\}$

K is a cutset of i and j if all paths connecting i and j intersect K.

That is, if 'K separates i and j'.

Cutsets and

Separation Graphoids and local CI-relations

$\langle G \rangle$ is a graphoid. A local relation $\mathcal{L} \in \mathscr{R}(N)$ is a *separation graphoid* if $\mathcal{L} = \langle G \rangle$ for some graph G; graph G is then unique.
If G is chordal or G is a forest it is a *chordal-separation graphoid* or *forest-separation graphoids*,respectively.

Contraction and Restriction Minors of Separation Graphoids local CI-relations

Lemma.

Minors of (chordal-)separation graphoids are (chordal-)separation graphoids.
The class of forest-separation graphoids is only contraction-closed.

The restriction of a forest-separation graphoid $\langle G \rangle$ by $k \in N$ is forest-separation iff there are at most 2 edges in G adjacent with k.

CI-relations over graphs

Separation Graphoids local CI-relations
Characteristic Minors

Proposition.

A ternary relation $\mathcal{I}$ is a separation graphoid iff each of its nontrivial 3-minors is isomorphic to one of the two graphoids of figure 6.

Figure 6. Nontrivial compulsory 3-minors of separation graphoids.

CI-relations over graphs

**Separation Graphoids local CI-relations
And
Pseudographoids**

Separation graphoids are obviously pseudographoids that are

Ascending *if*

$[(i,j|K) \in \mathcal{L}$ and $K \subset L \subset N - ij)] \Longrightarrow (i,j|L) \in \mathcal{L}$ and

complementary transitive *if*

$[(i,j|L) \notin \mathcal{L}$ and $(j,k|L) \notin \mathcal{L})] \Longrightarrow (i,k|L) \notin \mathcal{L}$:

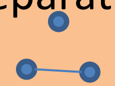
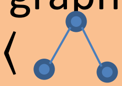
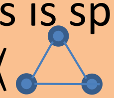
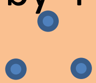
Content

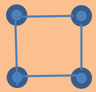
Chordal and Forest Separation Graphoids

CI-relations over graphs

Minors of Chordal Separation Graphoids

Consequence.

The class of chordal-separation graphoids is specified by 4 compulsory 3-minors ⟨  ⟩ , ⟨  ⟩ , ⟨  ⟩ and ⟨  ⟩

and 

one forbidden 4-minor .

A separation graphoid $\mathcal{L} \subset \mathcal{R}(N)$ is a chordal-separation graphoid iff it obeys

$$\{(i,j|klL),(k,l|ijL)\}\subset\mathcal{L} \Rightarrow \{(i,k|jlL),(i,l|jkL),(j,k|ilL),(j,l|ikL)\}\cap\mathcal{L} \neq \emptyset \text{ ;:}$$

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CI-relations over graphs

Chordal Separation Graphoids and pseudographoids

A relation $\mathcal{L} \subset \mathcal{R}(N)$ is a chordal-separation graphoid iff it is complementary transitive and ascending pseudographoid that satisfies

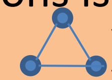
$$\{(i,j|klL),(k,l|*)\} \subset \mathcal{L} \Rightarrow \{(i,j|kL),(i,j|lL)\} \cap \mathcal{L} \neq \emptyset;$$

Forbidden minors of P-representability

CI-relations over graphs

Chordal Separation Graphoids that are Forest Separation Graphoids

Consequence.

A chordal-separation graphoid is forest-
separation iff none
of its contractions is isomorphic to
$\langle$  $\rangle$
or axiomatically, iff it satisfies

$$[(i,j|\ *)\notin\mathcal{L} \text{ and } (i,k|\ *)\notin\mathcal{L}] \Rightarrow (j,k\ |\ *)\in\mathcal{L}$$

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CI-relations over graphs

Forest Separation Graphoids local CI-relations

$\mathcal{L} \subset \mathcal{R}(N)$ is a forest-separation graphoid iff it is a complementary transitive and ascending pseudographoid that satisfies

$$[(i, j \mid *) \notin \mathcal{L} \text{ and } (i, k \mid *) \notin \mathcal{L}] \Rightarrow (j, k \mid i) \in \mathcal{L}$$

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Content

Forbidden minors of P-representability

Class of Boundary Semigraphoids

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Class of Boundary Semigraphoids

Having a simple undirected graph G,
the construction
$\langle G\rangle_{\text{loc}} = \{ (i, j|K) \in \mathcal{R}(N);\ K \supset \partial G_i$ or $K \supset \partial G_j\}$
is an ascending semigraphoid
The class of all semigraphoids of this type, called boundary semigraphoids, is not closed under restrictions.
Nevertheless, the class is closed under contractions
($\text{co}_{N-k} \langle G\rangle_{\text{loc}} = \langle G_{N-k}\rangle_{\text{loc}},\ k \in N$).

Class of Boundary Semigraphoids

Proposition.

A ternary relation $\mathcal{I} \subset \mathcal{R}(N)$ is a boundary semigraphoid iff $\mathcal{I}$ is ascending, satisfies the pseudographoid axiom for $\mathcal{I} = N - ijk$, and the following 3 implications:

$(i, j|kL) \in \mathcal{I}$, $(i, k|*) \in \mathcal{I}$ and $(j, k|*) \in \mathcal{I} \Rightarrow$
$(i, j|L) \in \mathcal{I}$, $(i, j|kL) \in \mathcal{I}$, $(i, k|*) \in \mathcal{I}$ and
$(j, l|*) \notin \mathcal{I}$, $l \notin ijkL \Rightarrow (i, j|L) \in L$, $(i, k|*) \notin \mathcal{I}$
and $(j, l|*) \notin \mathcal{I}$, $i \neq j \Rightarrow (i, j|N - ijkl) \notin \mathcal{I}$.

Forbidden minors of P-representability

Boundary semigraphoids are not necessarily graphoids

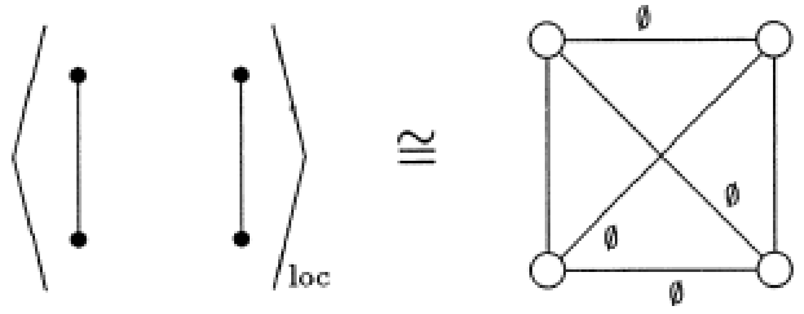

Figure 7. A boundary semigraphoid that is not a graphoid.

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Content

Forbidden minors of P-representability

d-separation Graphoids

S.L. Lauritzen, A.P. Dawid, B.N. Larsen and H.G. Leimer, Independence properties of directed Markov fields, Networks 20 (1990) 491–505. J. Pearl, Probabilistic Reasoning in Intelligent Systems (Morgan Kaufmann, San Mateo, CA, 1988).

d-separation Graphoids

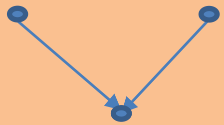
$\langle G \rangle_d$ = { $(i, j|K) \in \mathcal{R}(N)$; K separates i and j in $mor\ G_{an(ijK)}$ }

Here for $L \subset N$ the symbol $an(L)$ denotes
the set of all vertices of directed paths in G
that end in L, and
G_L is the vertex-induced subgraph as before.
(A set $L \subset N$ is ancestral if $L = an(L)$.)

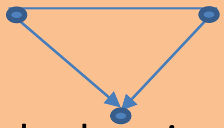
S.L. Lauritzen, A.P. Dawid, B.N. Larsen and H.G. Leimer, Independence properties of directed Markov fields, Networks 20 (1990) 491–505. J. Pearl, Probabilistic Reasoning in Intelligent Systems (Morgan Kaufmann, San Mateo, CA, 1988).

d-separation Graphoids obtained through moralization

The symbol mor, applied to a directed graph,
adds to every



its configuration the horizontal undirected edge



and converts all directed edges into the undirected ones.
This procedure is termed ***moralization***.
Every $\langle G \rangle_d$ is trivially a graphoid and every such graphoid is called a ***d-separation graphoid***.

S.L. Lauritzen, A.P. Dawid, B.N. Larsen and H.G. Leimer, Independence properties of directed Markov fields, Networks 20 (1990) 491–505. J. Pearl, Probabilistic Reasoning in Intelligent Systems (Morgan Kaufmann, San Mateo, CA, 1988).

Ancestrals sets of d-separation Graphoids

If i and j are arbitrary 2 different points of a path-with-tails G with the vertex set N, then ∃ exactly 1 vertex-induced subgraph of G that is a path-with-tails with the endpoints i and j and which has tails as long as possible.

We denote it by G^{ij} and its vertex set by N^{ij} .

if i (or j) was chosen in a tail of G the segment of that tail upwards of i (j) should be sloped appropriately and it ceases to be a tail in G^{ij} .

Tails of G^{ij} are tails of G otherwise.

A triple $(i,j|K) \in \mathcal{R}(N)$ does not belong to $\langle G \rangle_d$ iff $K \cap N^{ij}$ is covered by the tails of G^{ij} and contains a vertex of each tail of G^{ij}.

S.L. Lauritzen, A.P. Dawid, B.N. Larsen and H.G. Leimer, Independence properties of directed Markov fields, Networks 20 (1990) 491–505. J. Pearl, Probabilistic Reasoning in Intelligent Systems (Morgan Kaufmann, San Mateo, CA, 1988).

Ancestral sets of d-separation Graphoids

Lemma.

If $L = \langle G \rangle_d$ is a d-separation graphoid
and K is an ancestral set of G
then the restriction of L to K
and the contraction of L by K
are d-separation graphoids.

S.L. Lauritzen, A.P. Dawid, B.N. Larsen and H.G. Leimer, Independence properties of directed Markov fields, Networks 20 (1990) 491–505. J. Pearl, Probabilistic Reasoning in Intelligent Systems (Morgan Kaufmann, San Mateo, CA, 1988).

Forbidden minors of P-representability

d-separation Graphoids are not closed neither under restriction nor under contractions.

The class of d-separation graphoids is neither closed under restrictions nor under Contractions.
If a class of CI-relations is not minor-closed it can be often appealing to complete it by all its minors and look for the forbidden minors of this larger minor-closed class.
Minors can be interpreted as subconfigurations of causal structures, i.e., hidden causal structures

S.L. Lauritzen, A.P. Dawid, B.N. Larsen and H.G. Leimer, Independence properties of directed Markov fields, Networks 20 (1990) 491–505. J. Pearl, Probabilistic Reasoning in Intelligent Systems (Morgan Kaufmann, San Mateo, CA, 1988).