Probabilistic Conditional independence structures and Matroid Theory from F. Matus formalism

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ECE dept Fall 2016

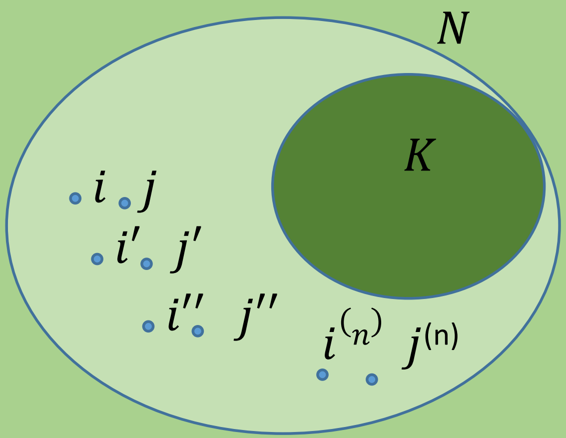
Forbidden minors of P-representability

Conditional Independence relations from subsets of a Finite set.

Probabilistic Conditional independence structures and Matroid Theory, F. Matus, Int. J. General Systems, Vol 22, pp. 185-196, 1994, Gordon and Breach Publishers

Content

Couples of elements out of
a subset of a Finite set.



Let N be a finite set,

Let $I, J, K \subset N$; $i, j, k \in N$ *(its singletons)*

$\mathfrak{S}(N)$ the family of all

$(i, j \mid K) = (a = i, b = j \mid K)$ s.t.

$\{(ij \mid K),\ K \subset N\ ,\ i, j \in N - K\}$

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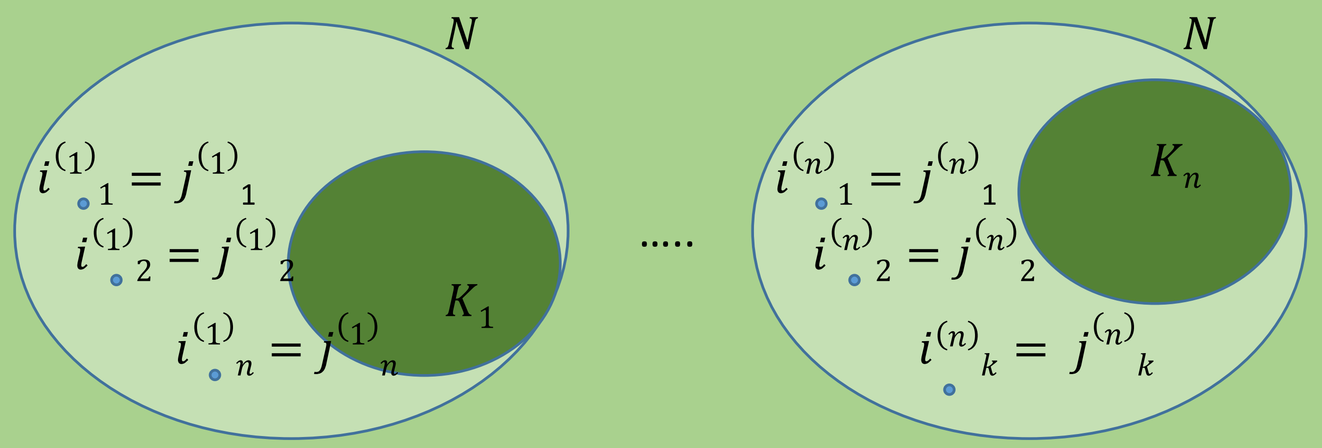
Classification of couples of elements out of a subset of a Finite set.

$\mathscr{Q}(N)$: are all the $(ij\,|K)$, where $i = j$

$\mathscr{R}(N)$: are all the $(ij\,|K)$, where $i \neq j$



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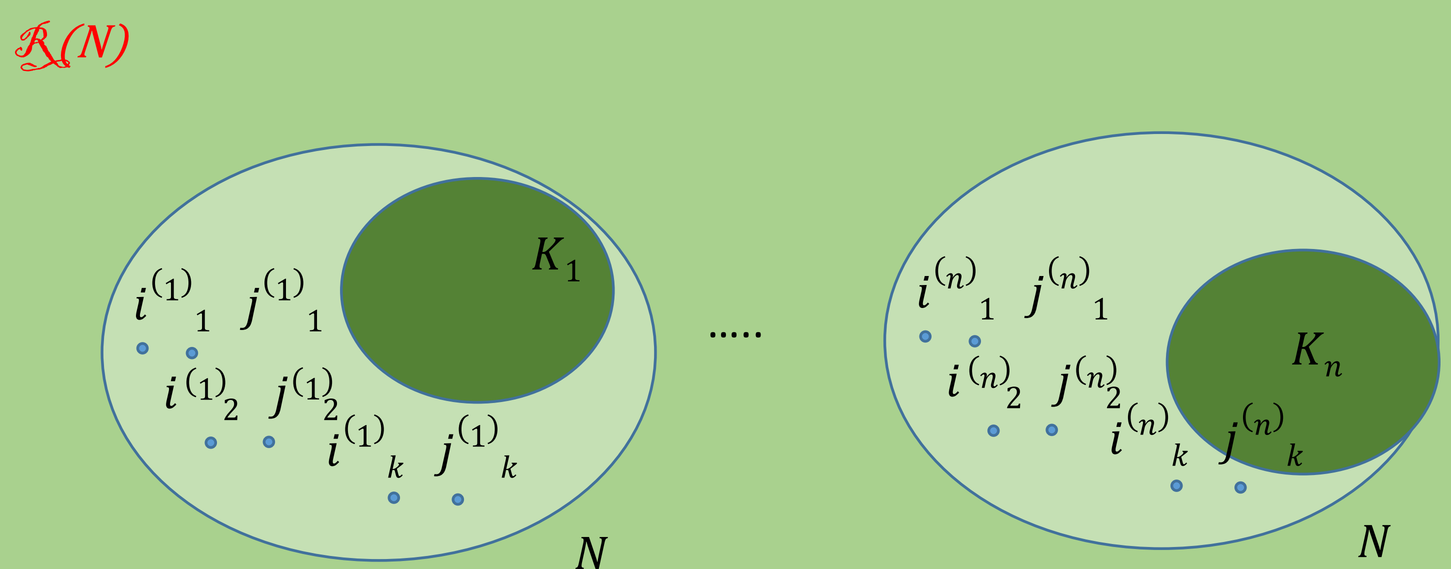
Local relations on a finite set *N*

$\mathcal{R}(N)$ are all the $(ij\,|K)$, where $i \neq j$



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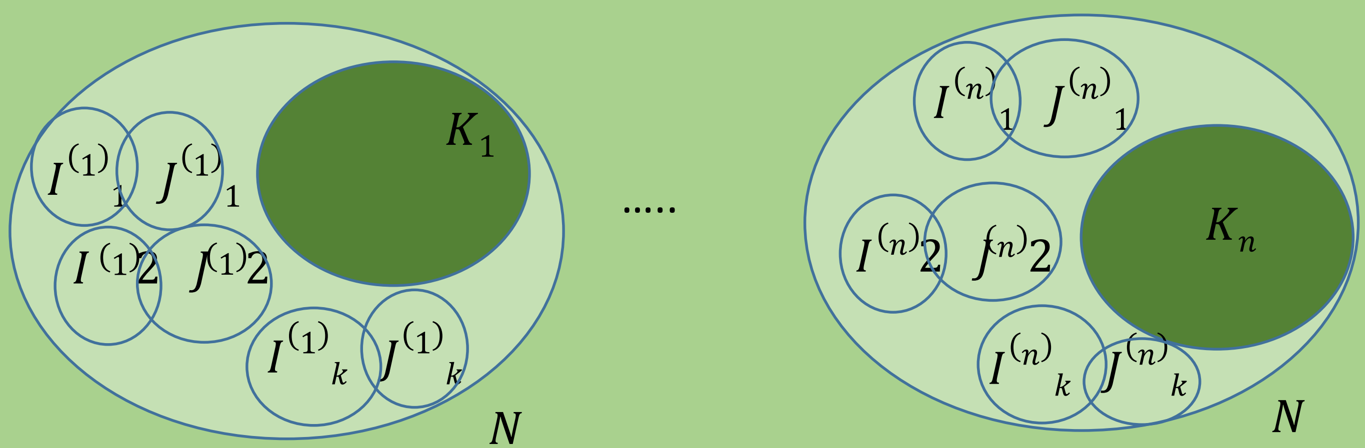
Global relations on a finite set *N*

$\mathcal{T}(N)$ are all the $(IJ \mid K)$, where $I, J, K \subset N$,

$\mathcal{T}(N)$



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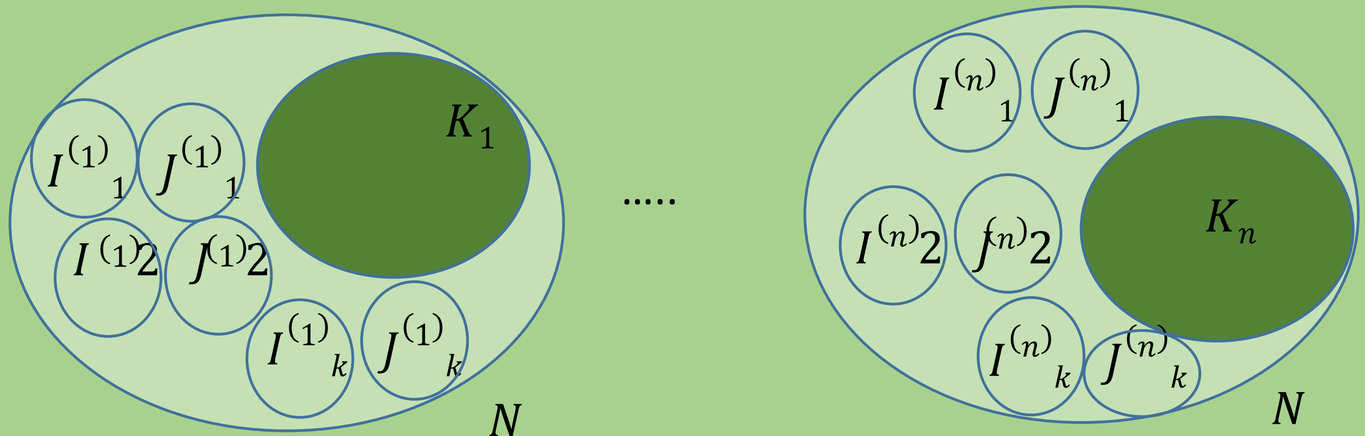
Global Conditional Independent relations on a finite set *N*

$\mathcal{J}(N)$ are all the $(IJ \mid K)$, where $I, J, K \subset N$, $s.t.\ I \cap J \cap K = \emptyset$

$\mathcal{J}(N)$



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Forbidden minors of P-representability

Example: CI relations on a finite set N

Let $N=\{n_1, n_2, n_3, n_4\}$

Then its index set is $I_N=\{1,2,3,4\}$

$\mathscr{P}(N)=\{\{n_1\}, \{n_2\}, \{n_3\}, \{n_4\}, \{n_1, n_1\},\{n_1, n_2\}, \{n_1, n_3\}, \{n_1, n_4\}, \{n_2, n_2\}\{n_2, n_3\}, \{n_2, n_4\}$
$, \{n_3, n_3\}, \{n_3, n_4\}, \{n_4, n_4\}, \},\{n_1, n_2, n_3\}, \{n_1, n_3, n_4\},\{n_1, n_2, n_4\}, \{n_2, n_3, n_4\}, \{n_1, n_2, n_3, n_4\}\}$

$\mathfrak{S}(N)=\{$ For $K_1= n_1$: 2,3|1; 2,4|1; 3,4|1; 2,2|1; 3,3|1; 4,4|1
For $K_2= n_2$: 1,3|2; 1,4|2; 3,4|2; 3,3|2; 1,1|2; 4,4|2
For $K_3= n_3$: 1,2|3; 1,4|3; 2,4|3; 1,1|3; 2,2|3; 4,4|3;
For $K_4= n_4$: 1,2|4; 1,3|4; 2,3|4; 1,1|4; 2,2|4; 3,3|4; }

$\mathscr{Q}(N)=\{$ For $K_1= n_1$: 2,2|1; 3,3|1; 4,4|1
For $K_2= n_2$: 3,3|2; 1,1|2; 4,4|2
For $K_3= n_3$: 1,1|3; 2,2|3; 4,4|3;
For $K_4= n_4$: 1,1|4; 2,2|4; 3,3|4; }

$\mathscr{R}(N)=\{$ For $K_1= n_1$: 2,3|1; 2,4|1; 3,4|1;
For $K_2= n_2$: 1,3|2; 1,4|2; 3,4|2
For $K_3= n_3$: 1,2|3; 1,4|3; 2,4|3;
For $K_4= n_4$: 1,2|4; 1,3|4; 2,3|4;}

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Forbidden minors of P-representability

Example: CI relations on a finite set *N*

Let $N=\{n_1, n_2, n_3, n_4\}$

$\mathscr{P}(N)=\{\{\}, \{n_1\}, \{n_2\}, \{n_3\}, \{n_4\}, \{n_1, n_1\}, \{n_1, n_2\}, \{n_1, n_3\}, \{n_1, n_4\}, \{n_2, n_2\}\{n_2, n_3\}, \{n_2, n_4\}$
$, \{n_3, n_3\}, \{n_3, n_4\}, \{n_4, n_4\}, \}, \{n_1, n_2, n_3\}, \{n_1, n_3, n_4\}, \{n_1, n_2, n_4\}, \{n_2, n_3, n_4\}, \{n_1, n_2, n_3, n_4\}\}$

$\mathscr{I}(N)=\{$ For $K_1= n_1$: 2,3|1; 2,4|1; 3,4|1; 2,2|1; 3,3|1; 4,4|1
For $K_2= n_2$: 1,3|2; 1,4|2; 3,4|2;
For $K_3= n_3$: 1,2|3; 1,4|3; 2,4|3;
For $K_4= n_4$: 1,2|4; 1,3|4; 2,3|4;
For $K_{12}=n_1, n_2$: {3},{4}|{1,2}; {},{4}|{1,2}; {3},{}|{1,2}; {},{3,4}|{1,2};
For $K_{13}=n_1, n_3$: {2},{4}|{1,3}; {},{2,4}|{1,3}; {},{4}|{1,3}; {},{2}|{1,3};
For $K_{14}=n_1, n_4$: {1},{3}|{1,4} ; {},{1,3}|{1,4}; {},{1}|{1,4}; {},{3}|{1,4};
For $K_8=n_2, n_3$: {1},{4}|{2,3}; {},{1,4}|{2,3}; {},{1}|{2,3}; {},{4}|{2,3};
For $K_{24}=n_2, n_4$: {1},{3}|{2,4}; {},{1,3}|{2,4}; {},{1}|{2,4}; {},{3}|{2,4};
For $K_{34}=n_3, n_4$: {1},{2}|{3,4}; {},{1,2}|{3,4}; {},{1}|{3,4}; {},{2}|{3,4};
For $K_{123}=n_1, n_2, n_3$: {},{4}|{1,2,3};
For $K_{134}=n_1, n_3, n_4$: {},{2}|{1,3,4};
For $K_{124}=n_1, n_2, n_4$: {},{3}|{1,2,4};
For $K_{234}=n_2, n_3, n_4$: {},{1}|{2,3,4};
For $K_{\emptyset}$ ={}: 1,3|{}; 1,4|{}; 3,4|{}; 1,2|{}; 1,4|{}; 2,4|{}; 1,2|{}; 1,3|{}; 2,3|{};
{3},{4}|{}; {2},{4}|{};
; {1},{4}|{};
{1,3},{2,4}|{}; {3,4},{1,2}|{}; {1,4},{2,3}|{}; {1,2,4},{3}|{}; {4},{1,2,3}|{}; {1,3,4},{2}|{}; {1},{2,3,4}|{}; }

Where in each of this triplets $a, b|c$ stands for $\xi_a, \xi_b|\xi_c$
And {} for $\xi_\emptyset$

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Content

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Probability distribution of Random variables
Of Matroids expressed in Conditional independence
Form

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Random variables for Conditional independence representation of Matroids

Random variables representing matroids
in their conditional independence
form are **uniformly distributed**.

This shifts the problem of probabilistically representable matroids
towards lattice theory, relating it with linear
and algebraic matroid representations.

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Quasi uniform distributed R.V. Systems and subsystems Indexed By a finite set N.

$$N = \{n_1, n_2, \dots, i, \dots, n_N\}$$

Let $\xi = (\xi_i)_{i \in N}$ be a system of r.v. indexed by N

$\xi_k = (\xi_i)_{i \in K}$, $K \subset N$ are its subsystems

$\xi_\emptyset$ is a constant

ξ_i take finite number of values

$$\xi_1 = \begin{cases} \Xi 1, \text{with pr}=1/p' \\ \dots \\ 0, \text{with pr}=1/m' \\ \dots \\ \Xi n', \text{with pr}=1/q' \end{cases} \quad \xi_2 = \begin{cases} \Xi 1'', \text{with pr}=1/p'' \\ \dots \\ 0, \text{with pr}=1/m'' \\ \dots \\ \Xi n'', \text{with pr}=1/q'' \end{cases} \quad \dots \quad \xi_k = \begin{cases} \Xi_1''', \text{with pr}=1/p''' \\ \dots \\ 0, \text{with pr}=1/m''' \\ \dots \\ \Xi_n''', \text{with pr}=1/q''' \end{cases}$$

$$m' + \dots + p' \dots + q' = n' \qquad m'' + \dots + p'' \dots + q'' = n'' \qquad m''' + \dots + p''' + \dots q''' = n'''$$

ASCENDING AND DESCENDING CONDITIONAL INDEPENDENCE RELATIONS, Frantisek Matus, Transactions of the Eleventh Prague Conference on Inform. Theory, Stat. Dec. Functions and Random Proc.ACADEMIA, Prague 1992, Vol 189-200, |

Systems and subsystems of R.V. indexed By a finite set N.

$\xi = (\xi_i)_{i \in N}$ a system of r.v.

$\xi_k = (\xi_i)_{i \in K}$, $K \subset N$ its subsystems

$\xi_\emptyset$ a constant

ξ_i take finite values

If we ignore just the zero values of the r.v.s we have:

$$\xi_1 = \begin{cases} \Xi_1, \text{with pr}=1/n' \\ \dots \\ \Xi_i, \text{with pr}=1/n' \\ \dots \\ \Xi_{n'}, \text{with pr}=1/n' \end{cases} \quad \xi_2 = \begin{cases} \Xi_{1''}, \text{with pr}=1/n'' \\ \dots \\ \Xi_{i''}, \text{with pr}=1/n'' \\ \dots \\ \Xi_{n''}, \text{with pr}=1/n'' \end{cases} \quad \dots \quad \xi_k = \begin{cases} \Xi_{1'''}, \text{with pr}=1/n''' \\ \dots \\ \Xi_{i'''}, \text{with pr}=1/n''' \\ \dots \\ \Xi_{n'''}, \text{with pr}=1/n''' \end{cases}$$

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Conditional Independence in Between r.v. indexed by sets

$$\xi : I \perp J \mid K,$$

$$I, J,\ K \subset N \Longrightarrow I, J,\ K \in \mathscr{P}(N)$$

stands for

ξ_I is conditionally independent of ξ_J given ξ_K

$$\Pr(\xi_I = \Xi_I \cap \xi_J = \Xi_J | \xi_K = \Xi_K)$$
$$= \Pr(\xi_I = \Xi_I, \xi_J = \Xi_J | \xi_K = \Xi_K) =$$
$$\Pr(\xi_I = \Xi_I | \xi_K = \Xi_K).\Pr(\xi_J = \Xi_J | \xi_K = \Xi_K)$$

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Conditional independence

The following two equivalences are valid for any ξ and $I, J, K, L \subset N$

$$\xi: I \perp J \mid K \Leftrightarrow \xi: J \perp I \mid K$$

$$\Pr(\xi_{I,}\xi_J|\xi_K) = \Pr(\xi_I|\xi_K).\Pr(\xi_J|\xi_K) = \Pr(\xi_J \cap \xi_I|\xi_K) = \Pr(\xi_{J,}\xi_I|\xi_K)$$

In simplified notation:

$$\xi: I \perp JK \mid L \Leftrightarrow \xi: I \perp J \mid KL \text{ and } \xi: I \perp K \mid L$$

In simplified notation:

$$\Pr(\xi_I\xi_{JK}|\xi_L) = \Pr(\xi_I|\xi_L).\Pr(\xi_{JK}|\xi_L);\ \Pr(\xi_I\xi_J|\xi_{KL}) = \Pr(\xi_I|\xi_{kL}).\Pr(\xi_J|\xi_{KL});$$

$$\Pr(\xi_I\xi_K|\xi_L) = \Pr(\xi_I|\xi_L).\Pr(\xi_K|\xi_L);$$

J. Appl. Prob, 29, 745-749, 1992, ON EQUIVALENCE OF MARKOV PROPERTIES OVER UNDIRECTED GRAPHS F. Matus

Conditional mutual Information

Let N be a finte set which
Elements index a system of r.v.s $\xi = (\xi_i)_{i \in N}$

For sets $I, J, K \subset N$

We have subsystems of r. v. s
$\xi_I = (\xi_i)_{i \in I}$; $\xi_J = (\xi_i)_{i \in J}$; $\xi_k = (\xi_i)_{i \in K}$
s.t. they attain values that
are approximately **uniformly distributed**
and can be enumerated with
The indices corresponding each of these sets

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Conditional mutual Information

Let N={$n_1, n_2, \ldots, n_N$} be a finite set

$1, 2, \ldots, N$

Is the index associated with its elements

Consider its subsets' indices

I={1,2,3,…, n'}

$J = \{n' + 1 = 1'', \ldots, n''\}$

$K = \{n'' + 1 = 1''', \ldots, N = n'''\}$

$$I, J, K \subset N \implies I, J, K \in \mathcal{P}(N)$$

We define r.v. with the same names

s.t.

$\xi_I = \{\Xi_I$, with pr$=1/n'\}$

$\xi_J = \{\Xi_J$, with pr$=1/n''\}$

$\xi_K = \{\Xi_K$, with pr$=1/n'''\}$

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Content

CI relations from subsets of a Finite set.
Probability dist. of R.v. of CI Matroids.
Shannon Entropy CI Description among R.V. subsystems.
Probabilistic CI via Globalizing Local CI relations.
Probabilistic Structures of Matroids and Semi-Matroids.
Conditional Relations and Connectedness
P- representable local and global relations
PDF of R.v.s of P- representable Local and global CI relations

Forbidden minors of P-representability

**Description of Stochastic Conditional Independence
Among Subsystems of
r.v.s
By Shannon Entropy**

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Conditional mutual Information

For $I, J, K \subset N$

Denote $h(I), h(J), h(K)$ a function on

These subsets

We define $\Delta\, h(I\,, J|\; K) = h(IK) + h(JK) - h(IJK) - h(K)$

$= h(I \cup K) + h(J \cup K) - h(I \cup J \cup K) - h(K)$

For any function h on $\mathcal{P}(N)$

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Class of functions $\mathcal{H}(N)$

a real valued function h on $\wp(N)$

We say $h \in \mathcal{H}(N)$ if

is locally non decreasing iff $\Delta\, h(i\,,i|\, K) \geq 0$

$(i,i|K\,) \in \mathcal{Q}(N)$.

$$h(i \cup K) - h(K) \geq 0$$

It is locally semi-modular iff $\Delta\, h(i\,,j|\, K) \geq 0$

$(i,j|K\,) \in \mathcal{R}(N)$.

$$h(i \cup K) + h(j \cup K) - h(i \cup j \cup K) - h(K) \geq 0$$

If also h is Normalized iff $h(\emptyset) = 0$

Then h is also globally non decreasing

Iff $\Delta\, h(I\,,I|\, K) \geq 0$; $I\,, K \subset N$

$$h(I \cup K) - h(K) \geq 0$$

and globally semi-modular

Iff $\Delta\, h(I\,,J|\, K) \geq 0$; $I\,, J, K \subset N$, disjoint

$$h(I \cup K) + h(J \cup K) - h(I \cup J \cup K) - h(K) \geq 0$$

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Entropy Balance Rule

For $I \subset N$

Let us denote h_ξ be the Shannon entropy function of

The system ξ_I

Then $\xi : I \perp J \mid K$

$\Pr(\xi_{I,} \xi_J | \xi_K) = \Pr(\xi_I | \xi_K) . \Pr(\xi_J | \xi_K)$

$\Pr(\xi_I = \Xi_{I,} \xi_J = \Xi_J | \xi_K = \Xi_K) = \Pr(\xi_I = \Xi_I | \xi_K = \Xi_K) . \Pr(\xi_J = \Xi_J | \xi_K = \Xi_K)$

Iff

We define $\Delta h_\xi(I , J | K) = 0$

$I, J, K \subset N$

$$h_\xi(I \cup K) + h_\xi(J \cup K) - h_\xi(I \cup J \cup K) - h_\xi(K) = 0$$

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Couples of elements outside of subsets of a Finite set characterize All Local conditional independence structures on it.

Given $h \subseteq \mathcal{H}(N)$ we are mainly interested in the local relations

$[h]$:{$(ij \mid K) \in \mathfrak{S}(N)$: $\Delta\, h(i, j \mid K)$ =0}

i.e.

$$h(\{i\} \cup K) + h(\{j\} \cup K) - h(\{i\} \cup \{j\} \cup K) - h(K) = 0$$

The entropy balance rule implies

$[h_\xi]$= $[\xi]$

The couples $\mathfrak{S}(N)$ are enough to capture

All the conditional independences

About subsystems of ξ

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Expressing Entropies of Global CI relations In terms of their corresponding Local CI structures on them.

Let $h \subseteq \mathcal{H}(N)$, if I={$i_1, \dots, i_s$}, $s \geq 1$, s.t. $I \cap K$=∅, $I, K \subset N$

Then $\Delta\, h(I\,, I|\; K) = \sum_{t=1}^{s} \Delta\, h(i_t\,, i_t|\; \{i_{\,t\,-\,1}, \dots, i_1\} \cup K)$

And if

J={$j_1, \dots, j_v$} $\subset N$-{$I \cup K$}, $v \geq 1$

Then

$$\Delta\, h(I\,, J|\; K) = \sum_{t=1}^{s} \sum_{u=1}^{v} \Delta\, h(i_t\,, i_t|\; \{i_{\,t\,-\,1}, \dots, i_1\} \cup \{j_{\,u\,-\,1}, \dots, j_1\} \cup K)$$

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Probabilistic Conditional independence
via
Globalizing Local CI relations .

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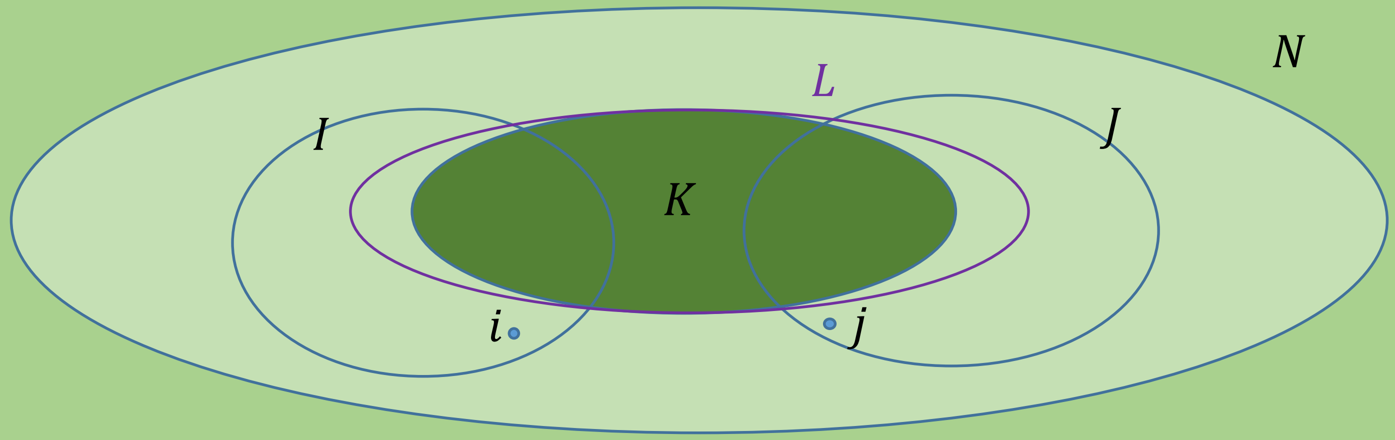
Globalizing Local conditional Relations .

Given a subset K of a finite set N , its Local CI relations can be extracted from corresponding Global CI relations among subsets of N w.r.t a set containing K.



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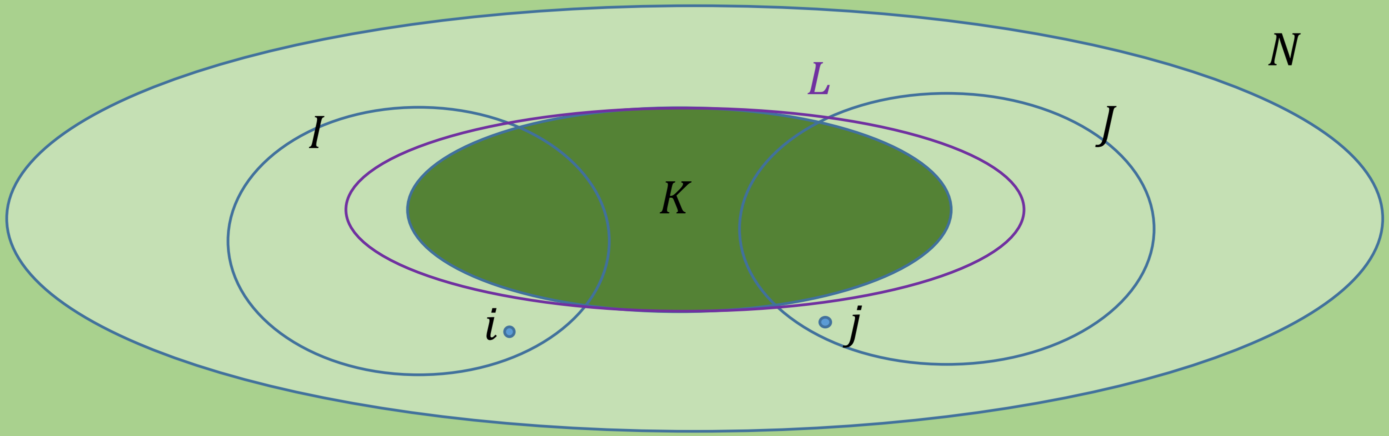
Global Operator for CI Local relations .

Let $\mathcal{L} \subset \mathfrak{S}(N)$;

$gl\,\mathcal{L} = \{I, J \mid K;\ I, J, K \subset N$

s.t.

$(\forall i \in I - K)(\forall j \in J - K)(\forall L \subset N)\ (K \subset L \subset IJK\text{-}ij \Longrightarrow i, j \mid L \subset \mathcal{L})\}$



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Global Operator for CI Local relations . Example

Let N={n_1, n_2, n_3, n_4}

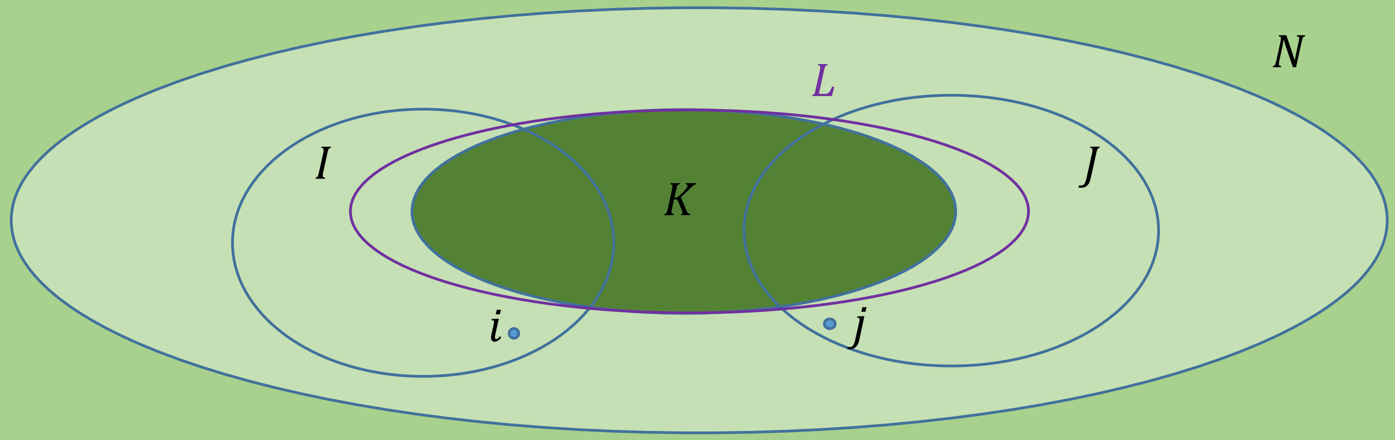
Then its index set is I_N={1,2,3,4}

Let $\mathcal{L} \subset \mathfrak{I}(N)$; $\mathcal{L}$={},{4}|{1}, here i ={},j={4},K ={1}

I={3,{}}; J={{},4};L = {1,2} ⟹ {} , 4|{1,2} ⊂ $\mathcal{L}$)}

$gl\,\mathcal{L}$ = {3,{}},{{},4}|{1,2}}

$(\forall i \in I - K)(\forall j \in J - K)(\forall L \subset N)$ ($K \subset L \subset \{I \cup J \cup K\}$-{{$i$} ∪ {$j$}} ⟹ i , j| $L \subset \mathcal{L}$)}



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Global CI is implied by Local CI .

For $h \subseteq \mathcal{H}(N)$ we have that $\Delta\, h(i\,, j|\; K) = 0$

Iff

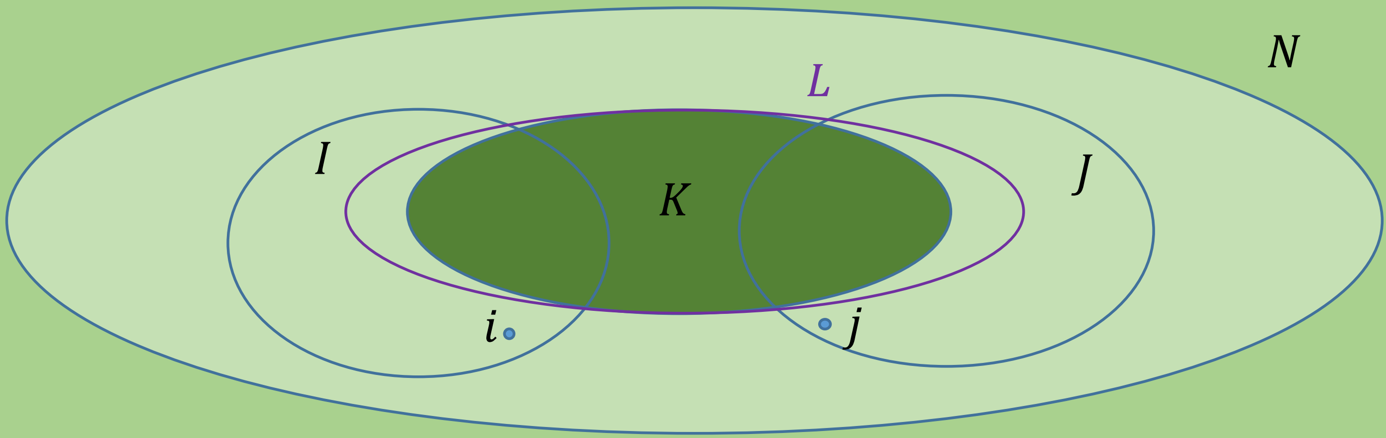
$(I\,, J|\; K) \in gl[h\,]$

Therefore

$\xi \colon I \perp J \;|\;\; K$

Iff

$(I\,, J|\; K) \in gl[\xi]$



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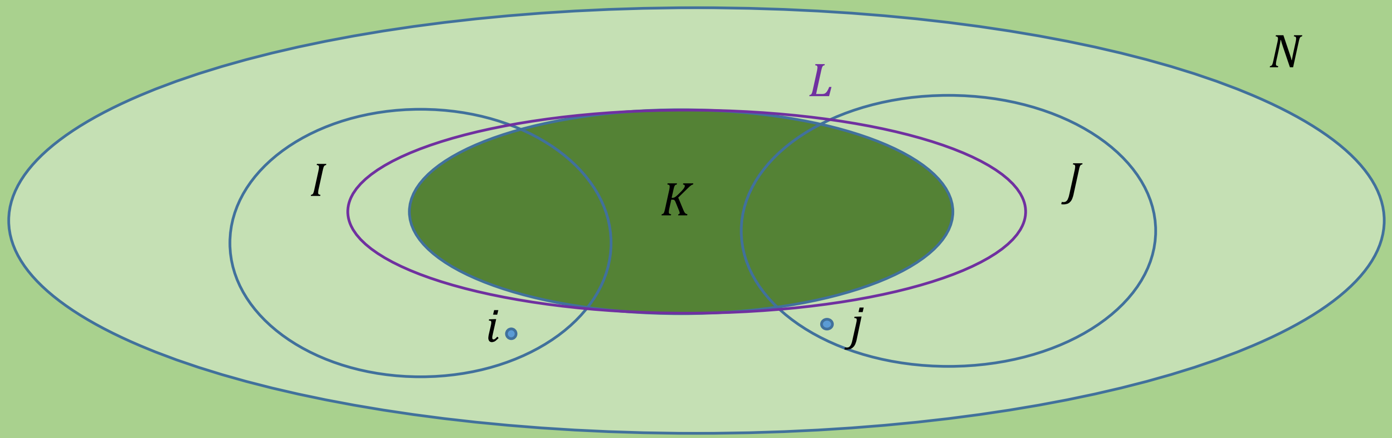
Global CI is implied by Local CI .

Lemma:

$\forall\ \xi$ and $I, J, K \subset N:\ [\xi: I \perp J \mid K] \Leftrightarrow (\forall i \in I)(\forall j \in J)(\forall L \subset N)\ (K \subset L \subset IJK{-}ij \Longrightarrow \xi: i \perp j \mid L)\}$



ON EQUIVALENCE OF MARKOV PROPERTIES
OVER UNDIRECTED GRAPHS, F. Matus, Int. Inf. Theory and Automation, Praga, J. Appl. Prob. 29: 745-749 (1992)

Lemma: (PROOF)

The implication ⇒ is consequence of the two properties:

1. $\xi : I \perp J \mid K \Leftrightarrow \xi : J \perp I \mid K$
2. $\xi : I \perp JK \mid L \Leftrightarrow \xi : I \perp J \mid KL$ and $\xi : I \perp K \mid L$

To prove ⇐ assume $I, J \neq \emptyset$ If $I = \{i\}, J = \{j\}$ $i, j \notin K$

right hand side gives $\xi : i \perp j \mid K$ if i or $j \in K$, left hand side holds trivially

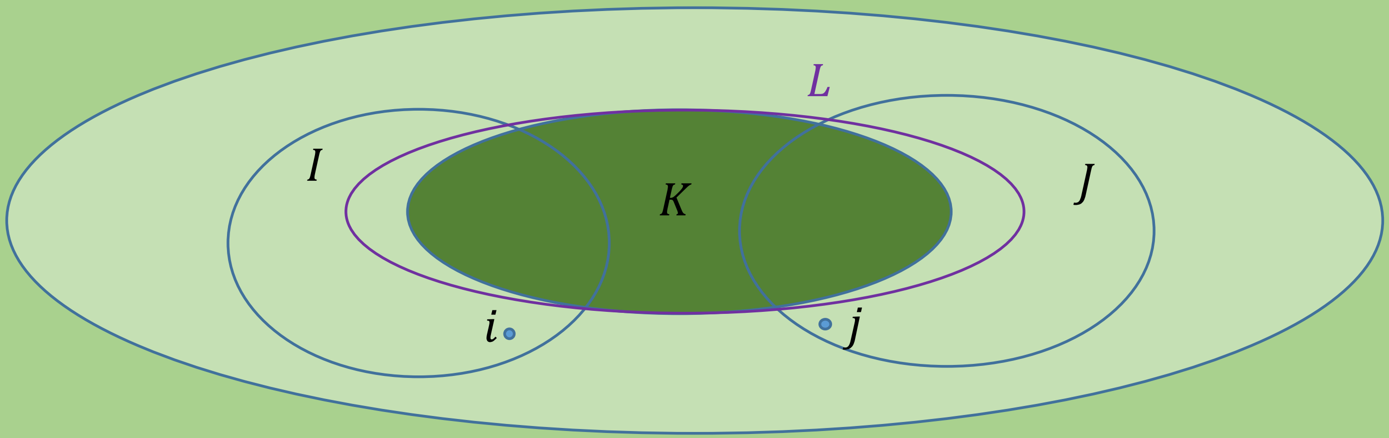
proceed by induction on $|I| + |J|$

Using symmetry of I , J decompose $J = J_1 \cup J_2, J_1 \cap J_2 = \emptyset$

By induction from right hand side $I \perp J_1 \mid K$ and $I \perp J \mid (J_1 \cup K)$

An application 2. closes the induction step.

Q.E.D



ON EQUIVALENCE OF MARKOV PROPERTIES
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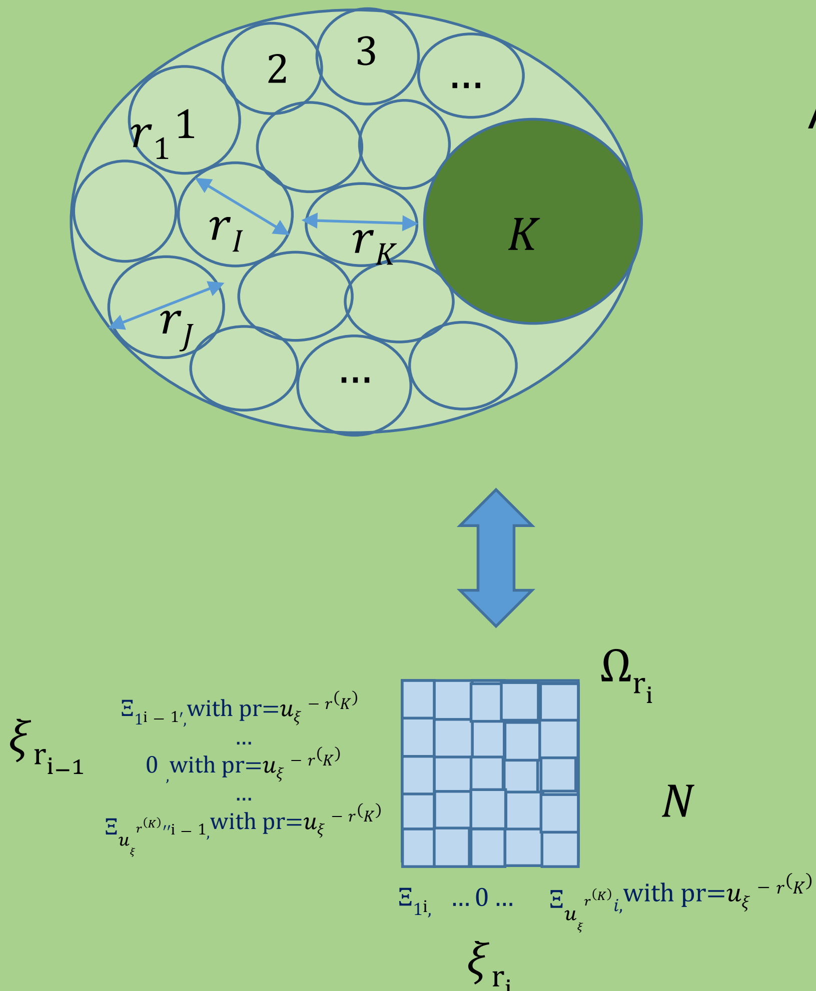
Content

Matroids as special classes of CI structures
and
Probabilistic structures as Semi-Matroids

Semi-Matroids



A relation $\mathcal{L} \subset \mathfrak{S}(N)$ is called Semi-Matroid
Iff
$\exists$ a function $r \in \mathcal{H}(N)$
s.t. $\mathcal{L}=[r]$

$[r]$:$\{(ij \mid K) \in \mathfrak{S}(N);\ \xi_r:\ i \perp j \mid K\}$

$\xi_r(i) \perp \xi_r(j) \ \mid \xi_r(K)$

$[r]$:$\{(ij \mid K) \in \mathfrak{S}(N): \Delta r(i, j \mid K) = 0\}$

All p-representable relations are
Due to the entropy balance rule
Semi-matroids

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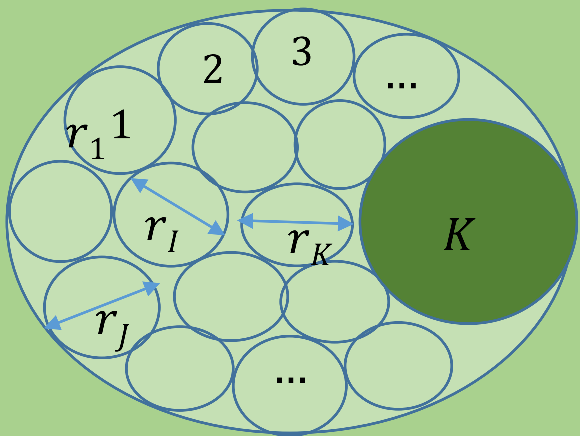
Intersection of Semi-Matroids

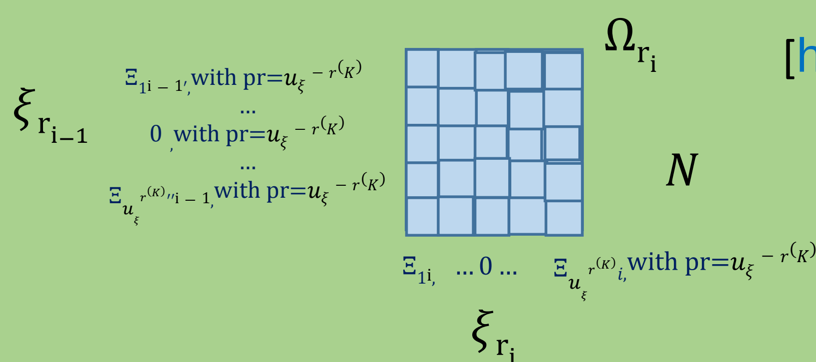


The intersection of 2 semimatroids $\mathcal{L}_s$=[h_s], s=1,2
Is the semimatroid $\mathcal{L}_1 \cap \mathcal{L}_2$=[$h_1$+ h_2]
then $\mathcal{L}_1 \cap \mathcal{L}_2 \subset \mathfrak{S}(N)$
Iff
$\exists$ a function $h_1+h_2 \in \mathcal{H}(N)$

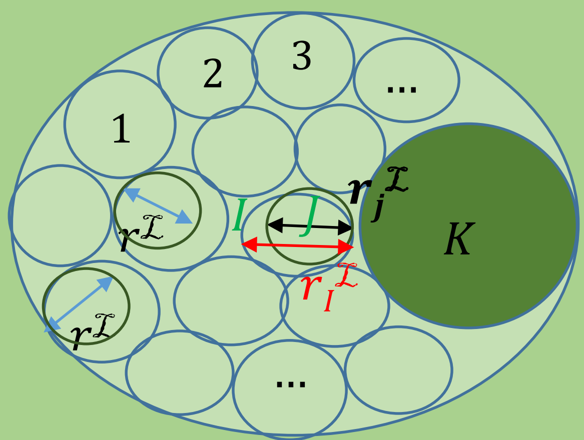
[h_1+ h_2] :$\{(ik \mid K) \in \mathfrak{S}(N);\ \xi_{h1+h2}:\ i \perp j \mid K\}$

$\xi_{h1+h2}(i) \perp \xi_{h1+h2}(j) \mid \xi_{h1+h2}(K)$

[h1+ h2] :$\{(ij \mid K) \in \mathfrak{S}(N)$: $\Delta h1+h2(i, j \mid K) = 0\}$

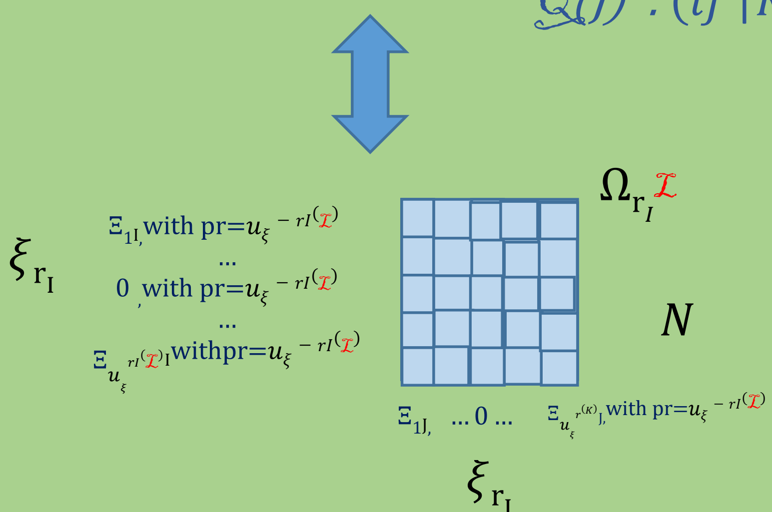


Matroids



The relation $\mathcal{L} \subset \mathfrak{S}(N)$ is called a matroid
iff
$r^{\mathcal{L}}$ =Max{$|J|$, $J \subset I$, $\mathfrak{Q}(J) \cap \mathcal{L} = \emptyset$} $I \subset N$
Is semimodular
and
$\mathcal{L}=[r^{\mathcal{L}}]$

$\mathfrak{Q}(J)\ : (ij\ |K), K \subset J, s.t.\ i = j$



Semi-modular refers to a type of lattice on which ∃ semimodular function $r^{\mathcal{L}}$ s.t. if J is a maximal element less than element I then $r^{\mathcal{L}}(J) + 1 = r^{\mathcal{L}}(I)$.

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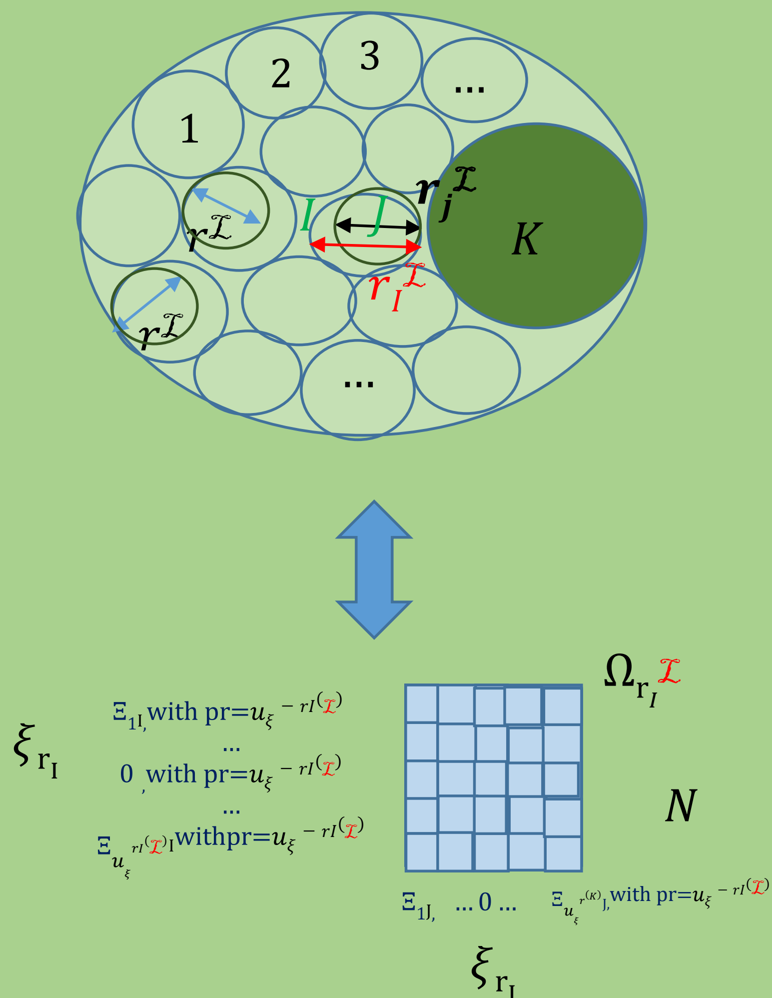
OVER UNDIRECTED GRAPHS, F. Matus, Int. Inf. Theory and Automation, Praga, J. Appl. Prob. 29: 745-749 (1992)

Matroids

$\mathcal{Q}(J)\ :(ij\ |K), K\subset J, s.t.\ i=j$



The relation $\mathcal{L}\subset\mathfrak{S}(N)$ is a matroid

iff

$r^{\mathcal{L}}$ =Max{$|J|$, $J\subset I$, $\mathcal{Q}(J)\cap\mathcal{L}=\emptyset$} $I\subset N$

Is semimodular and $\mathcal{L}=[r^{\mathcal{L}}]$

Proof:

The family of independent sets

$\mathcal{Y}_r=\{I\subset N, r(I)=I\}$;

$\emptyset\notin\mathcal{Y}_r$

$K\ \subset L\in\mathcal{Y}_r\Rightarrow K\in\mathcal{Y}_r$

$if\ (K,L\in\mathcal{Y}_r, |K|<|L|\,)\Rightarrow\exists\ i\in(L-K)s.t.$

$(iK\ \in\mathcal{Y}_r$ and $r(I)$ =Max{$|J|$, $J\subset I$, $J\in\mathcal{Y}_r$}

rank functions completely

specify Independent sets

Notice that if $I\in\mathcal{Y}_r\Leftrightarrow\mathcal{Q}(I)\cap[r]=\emptyset$

Iff $r=r^{\mathcal{L}}$ and then $\mathcal{L}$ is a matroid.

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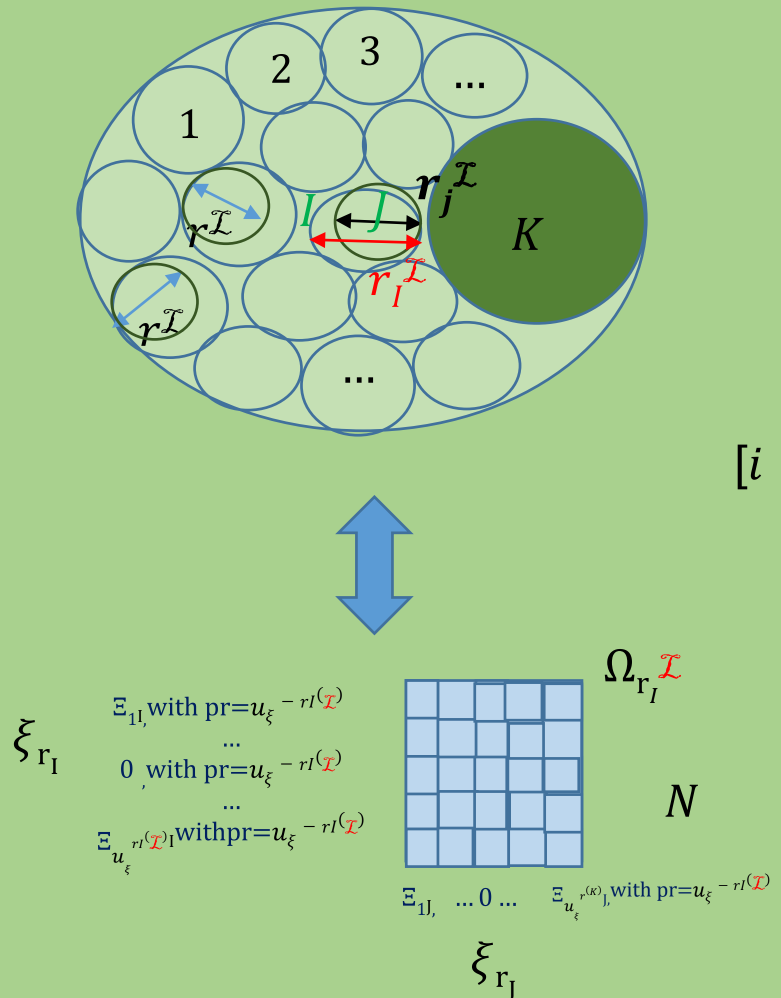
$\mathfrak{Q}(J)\ :(ij\ |K), K \subset J, s.t.\ i = j$
$\mathfrak{Q}(N)\ :(ij\ |K), s.t.\ i = j$



Matroids

The relation $\mathcal{L} \subset \mathfrak{S}(N)$ is a matroid

iff

$r^{\mathcal{L}}$=Max{$|J|$, $J \subset I$, $\mathfrak{Q}(J) \cap \mathcal{L} = \emptyset$} $I \subset N$

Is semimodular and $\mathcal{L}=[r^{\mathcal{L}}]$

Uniqueness of the Matroid:

Is unique through its dependence part

$\mathcal{L} \cap \mathfrak{Q}(I)$ that is

Transitive

$[i \notin J, (i|K) \in \mathcal{L}(\forall k \in K - J)((k|J) \in \mathcal{L})] \Rightarrow (i|J) \in \mathcal{L}$

Exchangable

$[\ (i|K) \in \mathfrak{Q}(N) - \mathcal{L}\ ((i|jK) \in \mathcal{L})] \Rightarrow (j|iK) \in \mathcal{L}$

$\therefore$ $\mathcal{L}$ is a Matroid Iff

$\mathcal{L}$ is Transitive, Exchangable and

$r = r^{\mathcal{L}}$.

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Forbidden minors of P-representability

Example:

Let $N = \{n_1, n_2, \ldots, n_N\}$
$I_N = \{1,2, \ldots, N\}$ be its index set
Consider indices of its subsets
$I_I = \{1,2,3, \ldots, n'\}$
$I_J = \{n' + 1 = 1'', \ldots., n''\}$
$I_K = \{n'' + 1 = 1''', \ldots., N = n'''\}$
$n', n'', n''' \in I_N$
$I, J, K \subset N \Longrightarrow I, J, K \in \mathscr{P}(N)$
I_I, I_J, I_K index subsets in Ω support of ξ
Define r.v. Indexed by them s.t.
$\xi_I = \{\Xi_I$, with pr$=1/n'\}$

$\xi_J = \{\Xi_J$, with pr$=1/n''\}$

$\xi_K = \{\Xi_K$, with pr$=1/n'''\}$

where Ξ_I, Ξ_J, Ξ_K are sequences of values
Indexed by *subsets* I_N

We have a Matroid if there exist
A function $r^{\mathcal{L}} \in \mathscr{H}(N)$
s.t.

$r^{\mathcal{L}} = \mathrm{Max}\{|J|, J \subset I, \mathcal{Q}(J) \cap \mathcal{L} = \emptyset\} \quad I \subset N$
Is semimodular

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Example:

Let $N = \{n_1, n_2, \ldots, n_N\}$
$I_N = \{1,2,\ldots, N\}$ be its index set
Consider indices of its elements
$I_i = \{i\}$ for singleton $\{n_i\}$
$I_j = \{j\}$ for singleton $\{n_j\}$
$I_k = \{k\}$ for singleton $\{n_k\}$
Each i, j or k is just a label
used to index singletons in the support Ω of ξ .
$i, j, k \subset N \implies \{i\}, \{j\}, \{k\} \in \mathscr{P}(N)$
We define r.v. with the same names
s.t.
$\xi_i = \{\Xi_i, \text{ with pr} = 1/n'\}$

$\xi_j = \{\Xi_j, \text{ with pr} = 1/n''\}$

$\xi_k = \{\Xi_k, \text{ with pr} = 1/n'''\}$

We have a semimatroid if there exist
A function $r^{\mathcal{I}} \in \mathcal{H}(N)$
s.t.

$[r] : \{(ij \mid K) \in \mathfrak{S}(N) : \Delta r(i, j \mid K) = 0\}$

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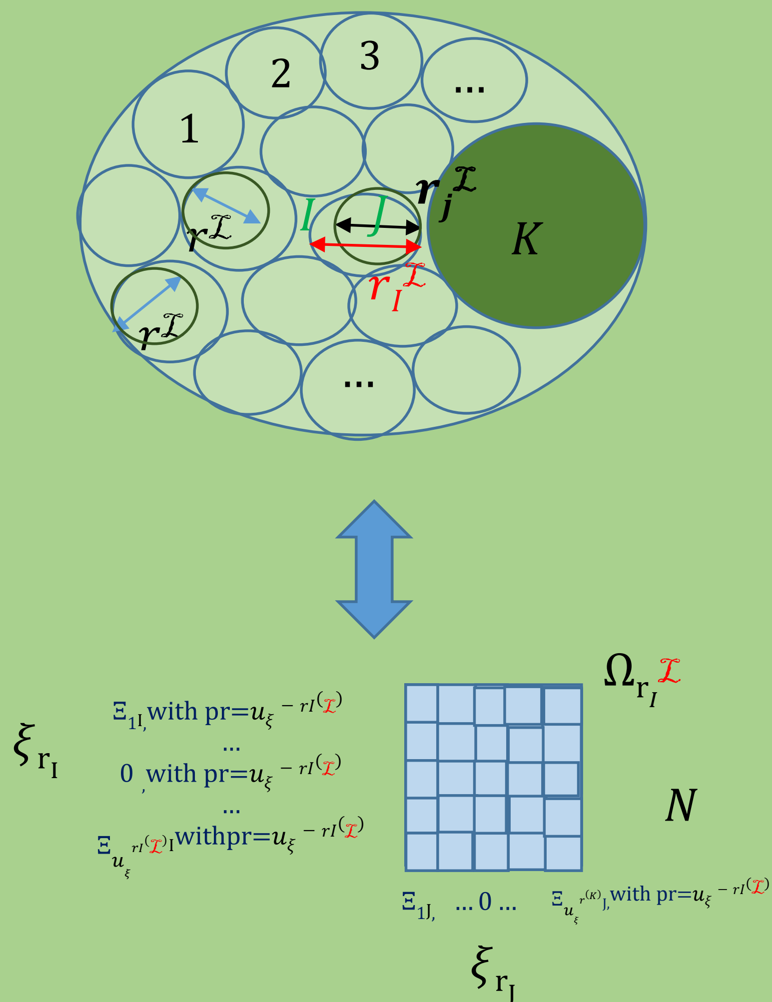
Matroid expansions of Semimatroids



Let $(M_i)_{i \in N}$ be a family of subsets

Of a finite set M

Let $M_I = \bigcup_{i \in I} M_i \quad I \subset N$

Construct for

$\forall\, g \in \mathcal{H}(M)$, an $h \in \mathcal{H}(N)$,

By $h(I) = g(MI)$

Then for $\forall$ semimatroid $\mathcal{K}=[g]$, we

Construct a semimatroid $\mathcal{L}=[h]$

By

$\mathcal{L}= \{\{ij|K) \in \mathcal{J}(N): (M_i, M_j | M_K) \in gl\, \mathcal{K}\}$

Then g and $\mathcal{K}$ is an expansion of

h and $\mathcal{L}$

$\mathcal{J}(N)$:all $(IJ\,|K): I, J, K \subset N,\ I \cap J \cap K = \emptyset$

ON EQUIVALENCE OF MARKOV PROPERTIES

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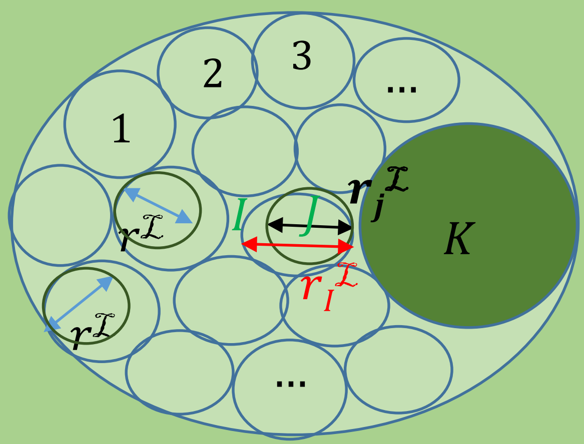
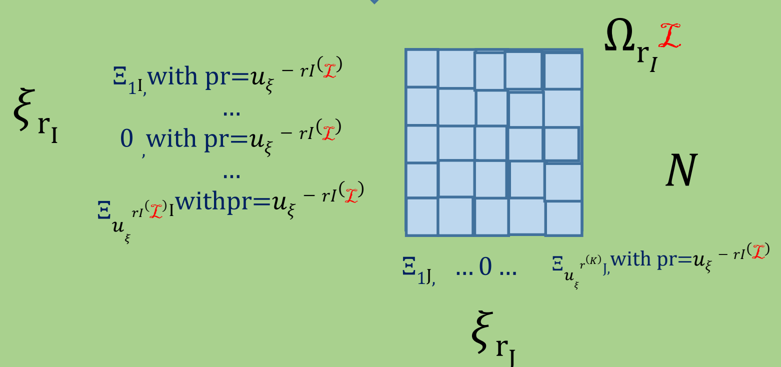
Matroid expansions of Semimatroids





Every semimatroid $\mathcal{L} \in \mathcal{S}(N)$ has a matroid expansion $\mathcal{K}$ *on a finite M partitioned in to* M_i, $i \in N$

Proof:

The class $\mathcal{H}(N)$ for a subset of $R^{\mathcal{P}(N)}$ is a Convex cone.

Since 1. $h \in \mathcal{H}(N)$ is positive $\forall\ h \neq 0$,

The cone is pointed defined by finite Number of inequalities , and an equality with finite Number of extreme rays.

Since $a_i x_i + \cdots + axn <> 0$

$b_i x_i + \cdots + b_n x_i = 0$ and $a_i, bi \in \mathbb{Z}$

ON EQUIVALENCE OF MARKOV PROPERTIES
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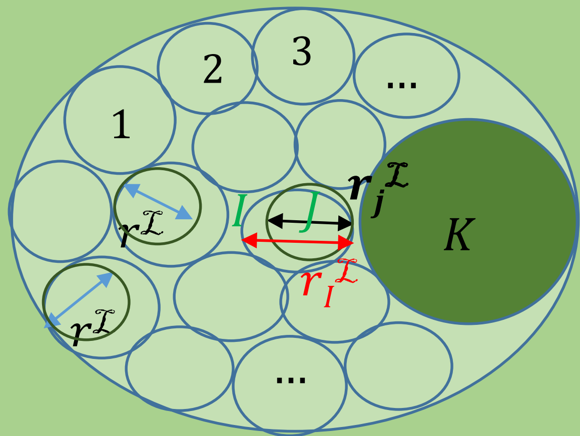
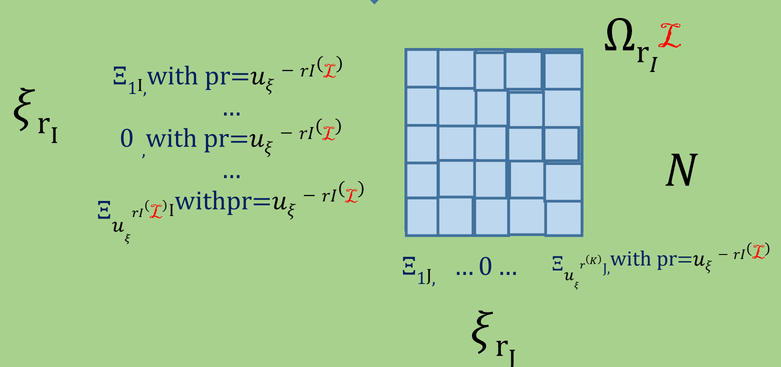




Every semimatroid $\mathcal{L} \in \mathcal{S}(N)$ has a matroid expansion

$\mathcal{K}$ *on a finite* M *partitioned in to* M_i, $i \in N$

Proof: (Cont.)

∃ an integer valued function on every extremal ray

If $\mathcal{L}$=[h] , $h \in \mathcal{H}(N)$ by Caratheodory Th.

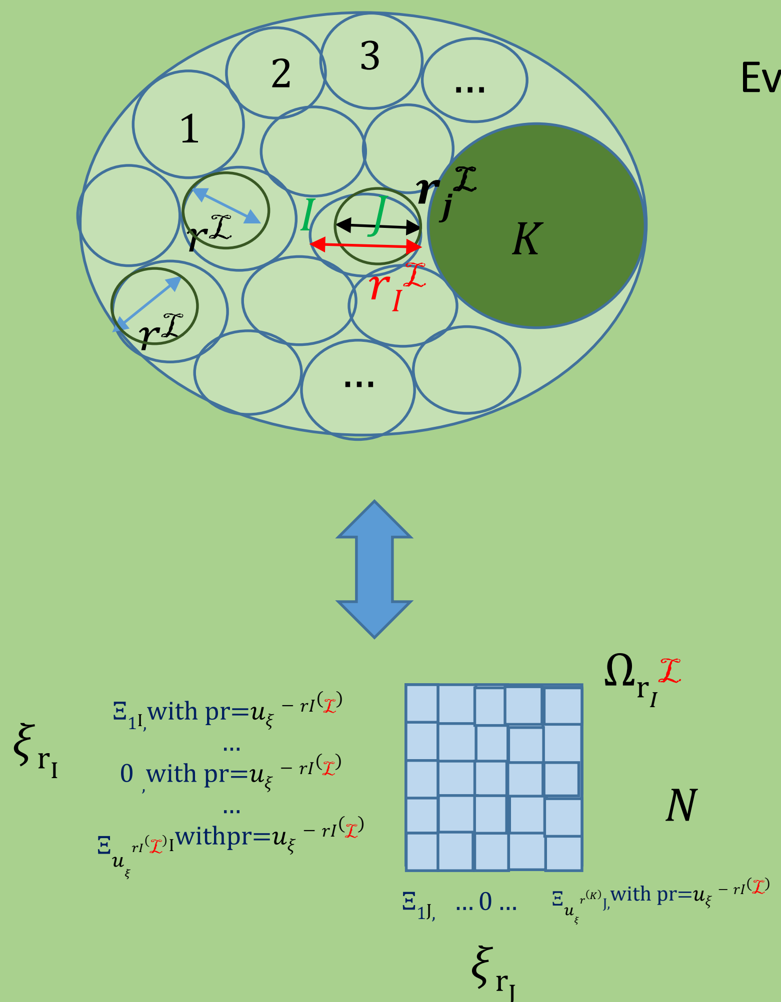
$h = \sum_{\alpha \in A} c_\alpha h_\alpha$ conical combination

Of integer valued h_α from extreme

Rays of $\mathcal{H}(N)$, $c_\alpha \in \mathbb{R}^+$

, $|A| \geq 2^{|N|}$. Let $\mathcal{L}=[h] = \bigcap_{\alpha \in A}[h_\alpha] = [\sum_{\alpha \in A} h_\alpha]$

Matroid expansions of Semimatroids

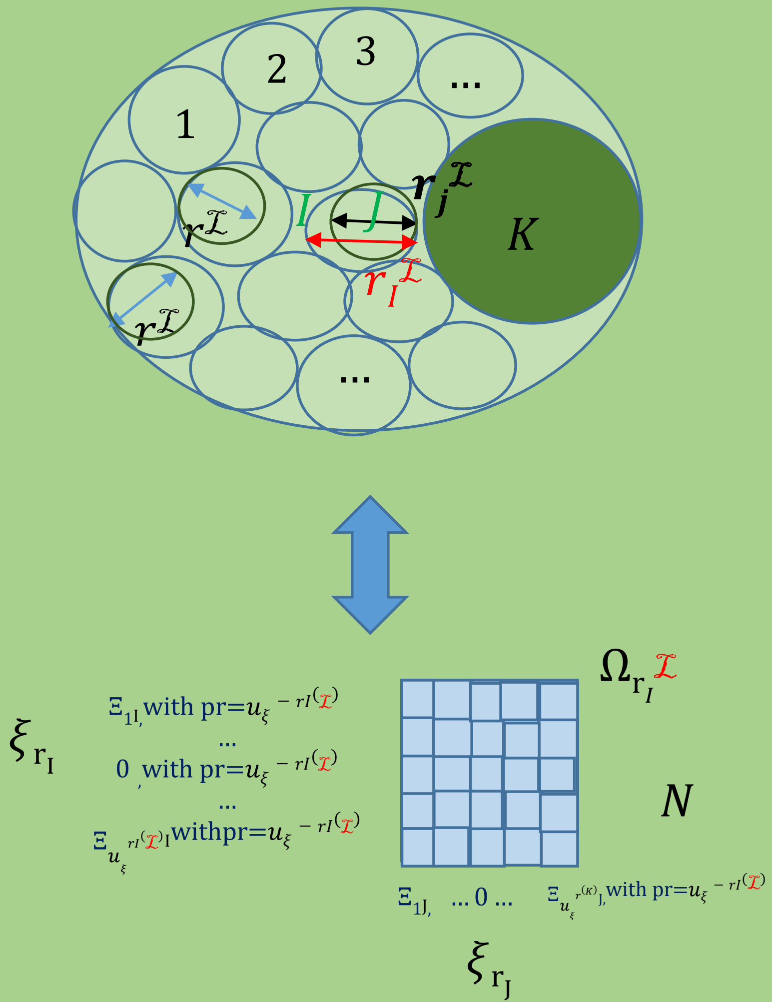


Every semimatroid $\mathcal{L} \in \mathcal{J}(N)$ has a matroid expansion $\mathcal{K}$ *on a finite* M *partitioned in to* M_i, $i \in N$

Proof (Cont.) :

Any integer valued $h \in \mathcal{H}(N)$
has a matroid expansion
r, i.e, there are finite disjoint M_i,
$i \in N$, $M_I = \sum_{i \in I} M_i$, for $I \subset N$
$M_N = M, Mi = h(i), i \in N$, together with
A rank function $r \in \mathcal{H}(M)$
s.t. $h(I) = r(M_i), I \subset N$
Thus $\mathcal{K}=[r]$ is a matroid expansion $\mathcal{L}=[h]$

Matroid expansions of Semimatroids



Obviously if an expansion $\mathcal{K}$ of a semimatroid
$\mathcal{L}$, is
$p-$ representable
$\eta = (\eta_j)_{j \in M}$
Then also $\mathcal{L}$ is $p-$ representable
$\xi = (\xi_i)_{i \in N}$
Where
$\xi_i == (\eta_j)_{j \in Mi}$

Open question:

Whether every p-representable semimatroid
Has a
p-representable matroid expansion.

Content

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P- representable local and global relations
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Forbidden minors of P-representability

Conditional Relations and Connectedness

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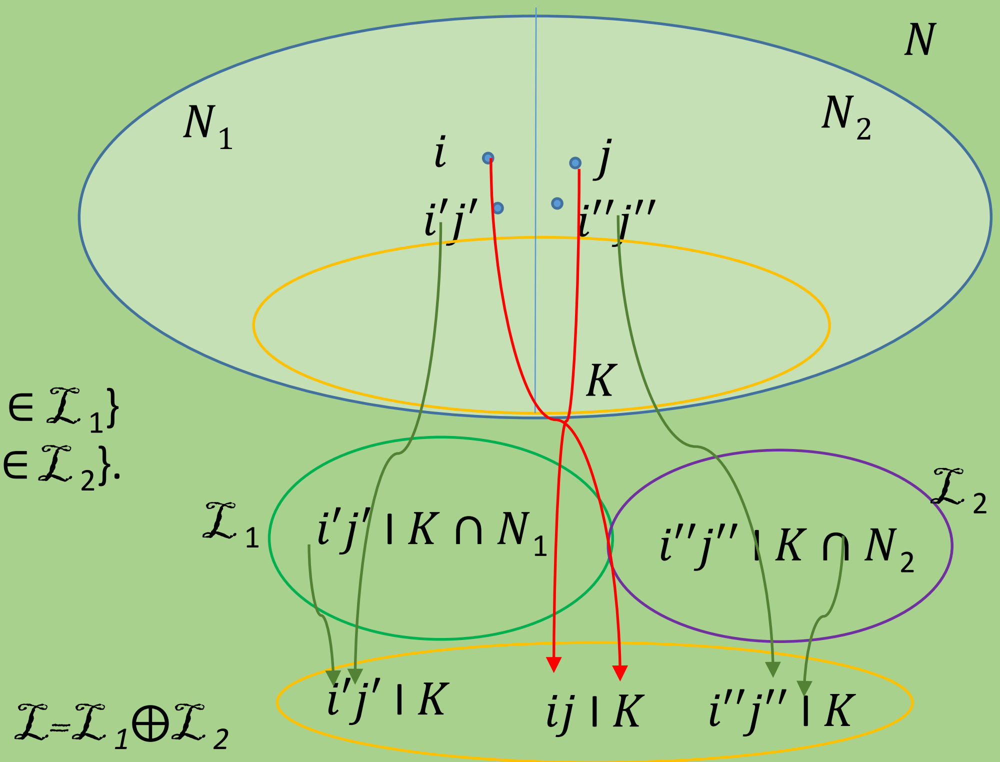
Forbidden minors of P-representability

Direct sum of Relations of elements couples Taken out of a Set

The *direct sum* of two relations
$\mathcal{L}_s \subset \mathfrak{S}(N_s)$; $s = 1, 2$,
$\mathcal{L}_1 \oplus \mathcal{L}_2$ I
on disjoint ground sets is
the relation $\mathcal{L}$ on $N = N_1 \cup N_2$
given by

$$\mathcal{L} = \{(ij \mid K) \in \mathfrak{S}(N);\ i \in N_1,\ j \in N_2\}$$
$$\cup \{(ij \mid K) \in \mathfrak{S}(N);\ ij \in N_1,\ (ij \mid K \cap N_1) \in \mathcal{L}_1\}$$
$$\cup \{(ij \mid K) \in \mathfrak{S}(N);\ ij \in N_2,\ (ij \mid K \cap N_2) \in \mathcal{L}_2\}.$$



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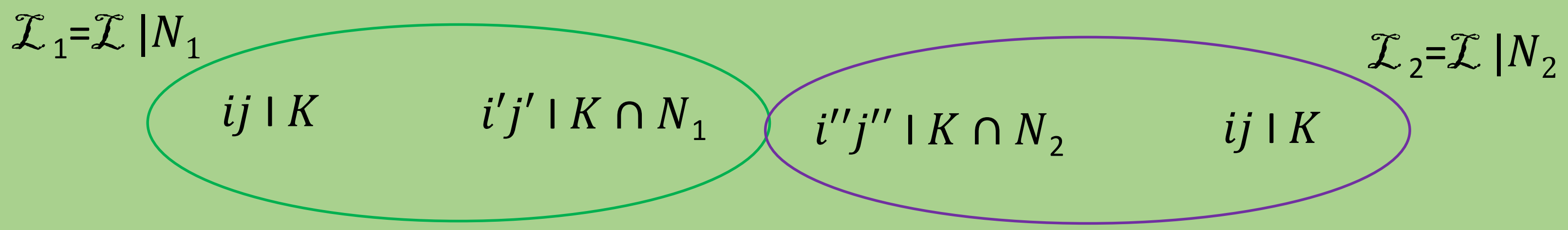
Direct sum and Connectedness in Semi-matroids



If a C.I. relation equates the direct sum of
Two C.I. relations that are basically restrictions
of it in to two disjoint subsets of the original finite set, for some particular
Partition of it, then we say it is a disconnected one, and the
Restrictions to the subsets are its connected components.

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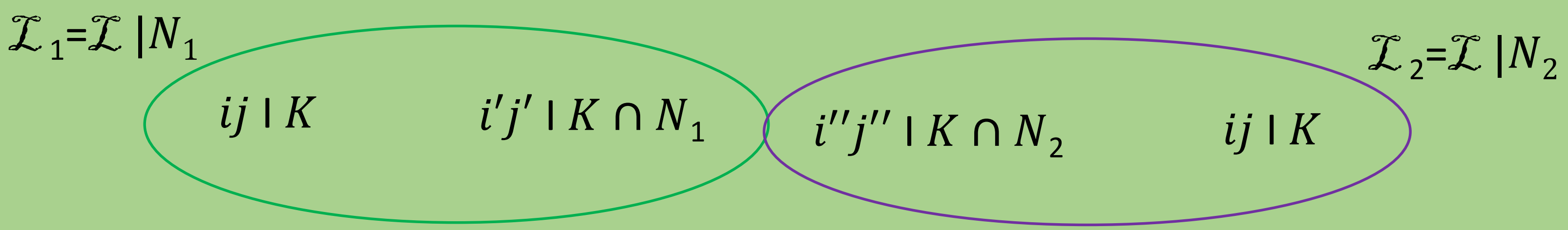
Direct sum and Connectness in Semi-matroids

Hence, $\mathcal{L}\,|N_s \;=\; \{(ij \mid K \cap N_s) \in \mathfrak{S}(Ns);\; (ij \mid K) \in \mathcal{L}\} = \mathcal{L}_s$,
$s \;=\; 1, 2$, and $(N_1\,, N_2 \mid \emptyset) \in gl\,\mathcal{L}$



($\mathcal{L}$ is the greatest relation on N with these properties).
If $\mathcal{L}_s$ is a matroid with the rank function r_s, $s \;=\; 1, 2$, then $\mathcal{L}$ is the matroid with the rank function $r(I) = r_1(I \cap N_1) + r_2(I \cap N_2)$, $I \subset N$, and our definition of the direct sum coincides with the standard one.

The direct sum of semimatroids is a semimatroid.

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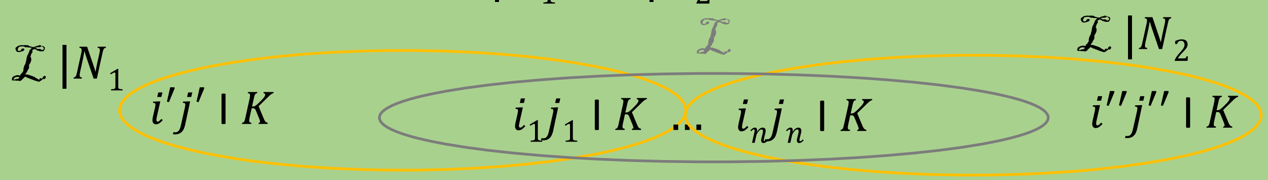
Forbidden minors of P-representability

Disconnected set Relations.

Given relations $\mathcal{L}\,|N_s$, s = 1, 2, For $N_1 = \emptyset$ or $N_2 = \emptyset$*,* then

$$\mathcal{L}\,|N_1 \oplus \mathcal{L}\,|N_2 \supseteq \mathcal{L}$$



If this direct sum coincides with $\mathcal{L}$ for a nontrivial partition then the relation $\mathcal{L}$ is called *disconnected.*

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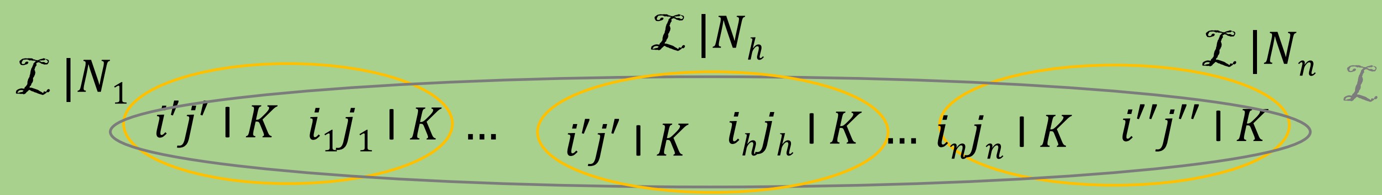
Connected Components of matroids and Semimatroids.

∀ relation $\mathcal{L}$, ∃ a unique partition of N into nonempty blocks N,

s.t.

$\mathcal{L} = \oplus\mathcal{L}\,|N_t$ t = 1, 2, … , s, s ≥1.



connected relations (components)

The connected components of

matroids and semimatroids

are

matroids and semimatroids, respectively.

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P- representable local and global relations

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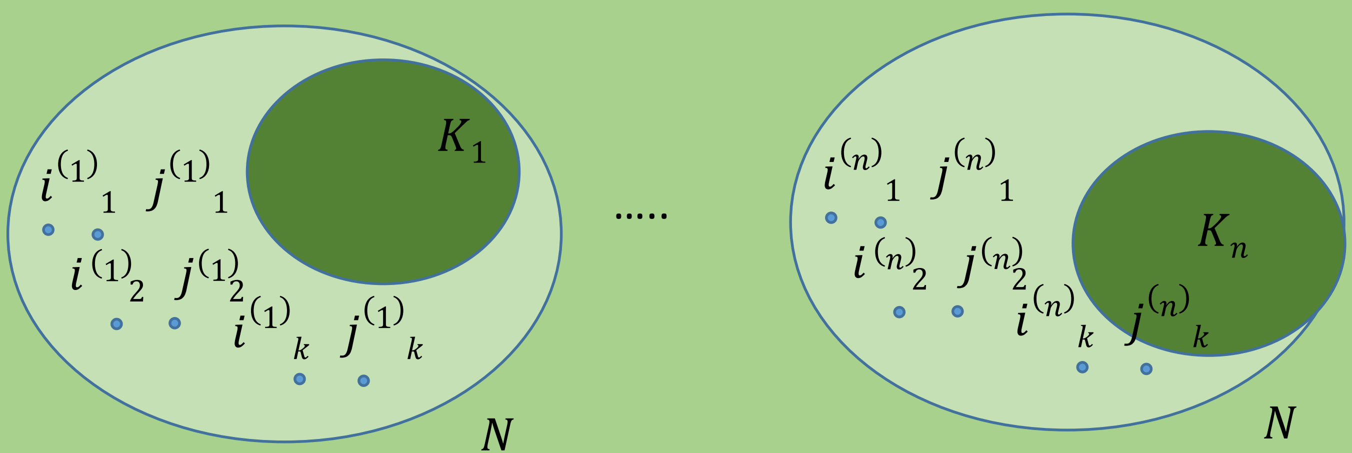
Forbidden minors of P-representability

P-representable C.I. relations

If there exist partitions of a finite set that
Determine global or local C.I. relations on it,
we say that they correspond to random variables
That are partition representable .



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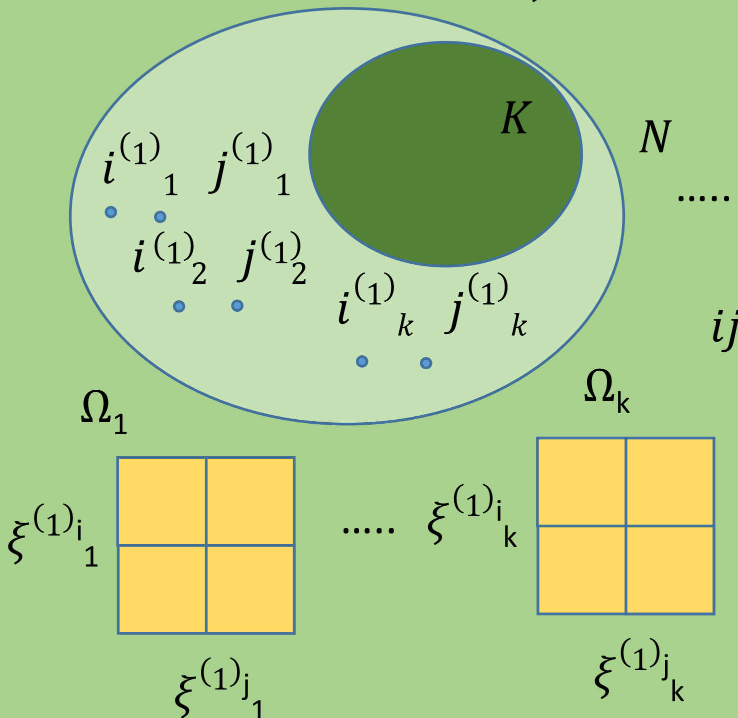
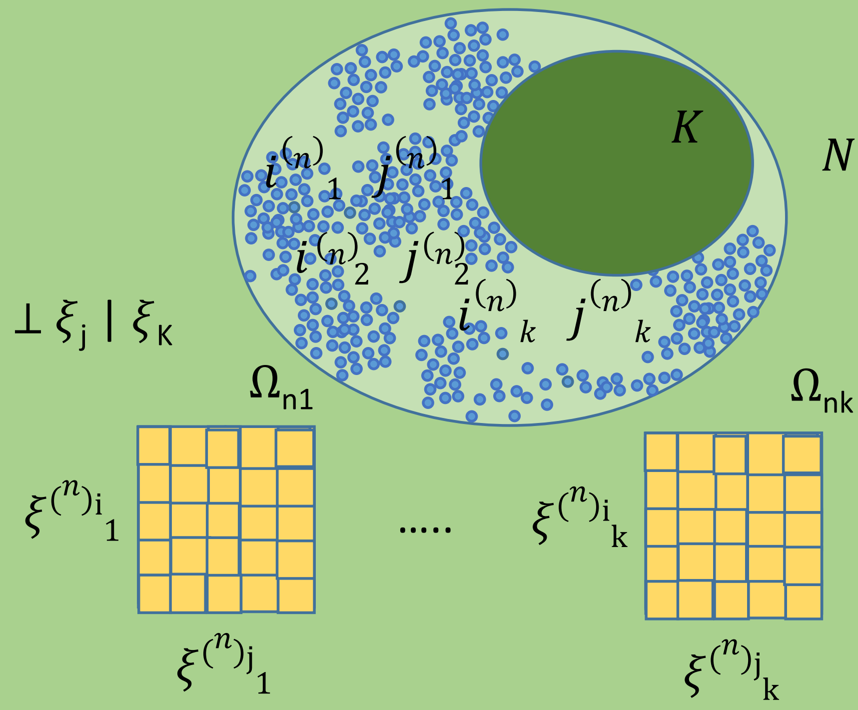
Forbidden minors of P-representability

***P*-representable local relations**

A local relation $r \subseteq \mathcal{R}(N)$ is p-representable if
$\exists\, \xi$ s.t. $r = [\xi] : \{(ij \mid K) \in \mathcal{R}(N);\ \xi :\ i \perp j \mid K\}$



s.t.
$ij \mid K \in r$ iff $\xi_i \perp \xi_j \mid \xi_K$



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P-representable Global relations

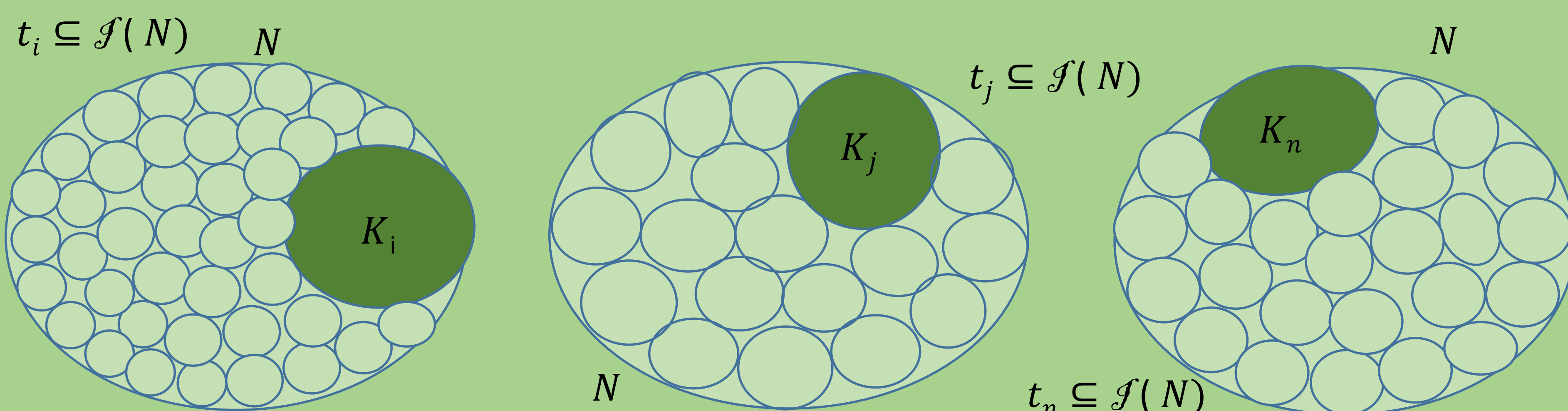
A global relation $t \subseteq \mathcal{J}(N)$ is p-representable if

$\exists\, \xi$ s.t. $t = [\xi] : \{(IJ \mid K) \in \mathfrak{J}(N);\ \xi : I \perp J \mid K\}$

⇕

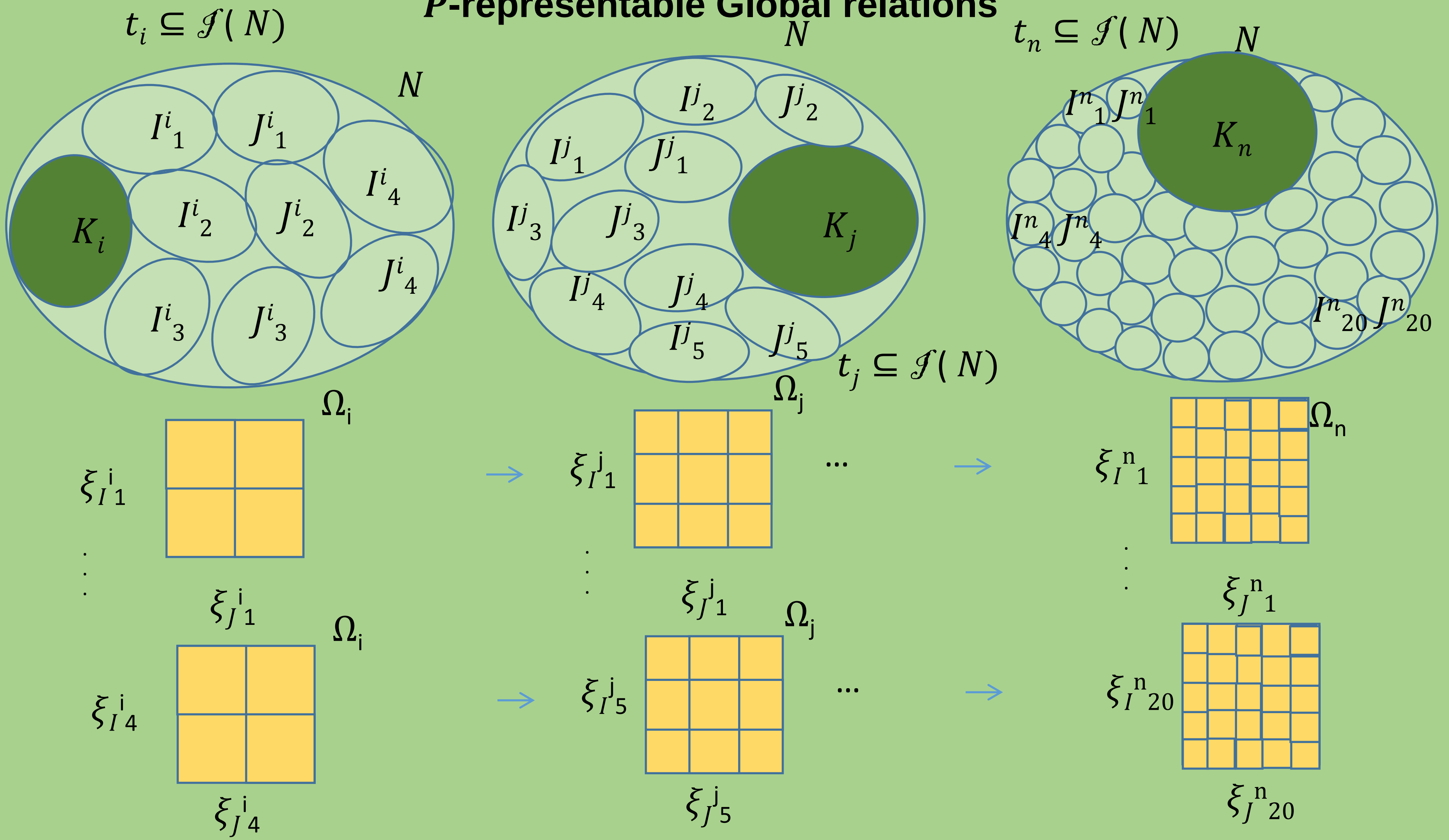
s.t.

$IJ \mid K \in t$ iff $\xi_I \perp \xi_J \mid \xi_K$



ASCENDING AND DESCENDING CONDITIONAL INDEPENDENCE RELATIONS, Frantisek Matus, Transactions of the Eleventh Prague Conference on Inform. Theory, Stat. Dec. Functions and Random Proc.ACADEMIA, Prague 1992, Vol 189-200,

P-representable Global relations



ASCENDING AND DESCENDING CONDITIONAL INDEPENDENCE RELATIONS, Frantisek Matus, Transactions of the Eleventh Prague Conference on Inform. Theory, Stat. Dec. Functions and Random Proc.ACADEMIA, Prague 1992, Vol 189-200,

Content

CI relations from subsets of a Finite set.
Probability dist. of R.v. of CI Matroids.
Shannon Entropy CI Description among R.V. subsystems.
Probabilistic CI via Globalizing Local CI relations.
Probabilistic Structures of Matroids and Semi-Matroids.
Conditional Relations and Connectedness
P- representable local and global relations
PDF of R.v.s of P- representable Local and global CI relations

Forbidden minors of P-representability

Probability Distributions of r.v.s of
***P*- representable**
local and global CI relations

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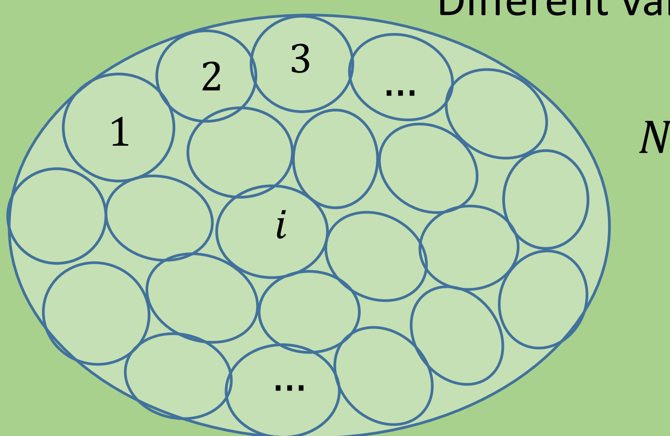
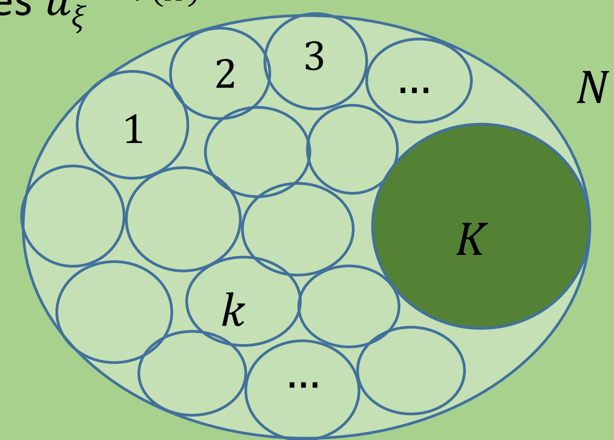
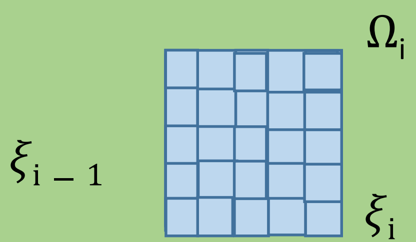
P −representation of connected Matroids

THEOREM

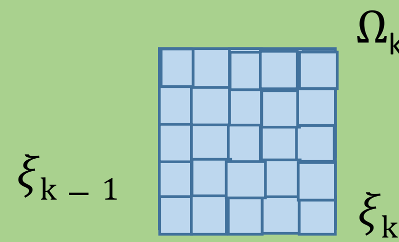
Let If $\xi = (\xi_i)_{i \in N}$ be a p −representation of a connected matroid $\mathcal{L}$ with the rank function r and $r(N) \geq 2$. Then ∃ a unique integer $u_{\xi} \geq 2$, we shall call it *degree* of ξ, s.t. ∀ subsystem ξ_K, $K \subset N$, takes just $u_{\xi}^{r(K)}$ Different values with the same probabilities $u_{\xi}^{-r(K)}$







$$\xi_K = \begin{cases} \Xi_{1'}, \text{with pr} = u_{\xi}^{-r(I)} \\ \ldots \\ \Xi_{i'}, \text{with pr} = u_{\xi}^{-r(I)} \\ \ldots \\ \Xi_{u_{\xi}^{r(I)}{}''}, \text{with pr} = u_{\xi}^{-r(I)} \end{cases} \qquad \xi_K = \begin{cases} \Xi_{1''}, \text{with pr} = u_{\xi}^{-r(K)} \\ \ldots \\ \Xi_{k''}, \text{with pr} = u_{\xi}^{-r(K)} \\ \ldots \\ \Xi_{u_{\xi}^{r(K)}{}''}, \text{with pr} = u_{\xi}^{-r(K)} \end{cases}$$



Probabilistic Conditional independence structures and Matroid Theory, F. Matus, Int. J. General Systems, Vol 22, pp. 185-196, 1994, Gordon and Breach Publishers

P −representation of connected Matroids

THEOREM (Proof)

Let ξ =(ξ_i)$_{i\ \in\ N}$ be r.v. taking values on a set X,
∀ outcomes Pr(ξ_i=x_i), $x_i \in X$, are positive, then values x_k of a subsystem ξ_k
$K \subset N$, are in the cartesian product X_K=$\sum_{k \in K} X_k$
Let $i \neq j \in N$ then since there are no loops, we have :

$$N = \{1,2,3, \dots, n\}$$
$$\xi = (\xi_1 = x_1, \dots \xi_i = xi, \dots \xi_n = xn)_{i\ \in\ N}$$
$$K = \{k - b, \dots k\} \in N,$$
$$\xi_k = (\xi_i = xk_{-b}, \dots .\ \xi_k = x_k) \in \sum_{k \in K} X_k$$

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P −representation of connected Matroids

THEOREM (Proof)

Case1:

$r(ij) = 1$, they are parallel,
then $(i|j), (j|i) \in \mathcal{L}$,
so ξ_i is a function of ξ_j
and viceversa,
thus $|X_j|$=$|X_i|$,
Therefore distributions ξ_i and ξ_j
coincide up to a
bijection of X_i to X_j .

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P −representation of connected Matroids

THEOREM (Proof)

Case2: $r(ij) = 2$, set i, j as independent , then
∃ circuit $L \subset N$ of the matroid $\mathcal{L}$ s.t. $i, j \in \mathcal{L}$,
Let $K = L - ij \neq \emptyset$, since $\forall\, xi \in X_i, x_j \in X_j$,
$(x_i\, x_j)$=Pr($\xi_i = x_i\, \xi_j = x_j$)=Pr($\xi_i = x_i$)Pr($\xi_j = x_j$)> 0
Then ∃ $x_k \in X_k$ s.t. $(x_i\, x_k\, x_j) \geq 0$ and $(x_k) \geq 0$

According with $(i, K|\emptyset)$ and $(j, K|\emptyset) \in gl(\mathcal{L})$and $(i, jK|\emptyset)$ and $(j, iK|\emptyset) \in \mathcal{L}$,
$(i, jK|\emptyset)$=$(i, j \cup K|\emptyset)$; $(j, iK|\emptyset)$=$(j, i \cup K|\emptyset)$
then
$(x_i)\ (x_k)$=$(x_i\, x_k)$= $(x_i\, x_k\, x_j)$= $(x_j\, x_k)$= $(x_j)\ (x_k)$
$\xi_i$and ξ_j are unif. Distrib. and $|X_j|$=$|X_i|$,
Then ξ_i are unif. Distrib. over sets of same cardinality
$u_{\xi} \geq 2$.

P −representation of connected Matroids

THEOREM (Proof)

If I is not independent set of $\mathcal{L}$, then
all ξ_I are mutually independent , so unif. Distrib. on a set of cardinality $u_{\xi}{}^{I}$.
If we choose $\emptyset \neq K \subset N$,
let I be one of its maximal independent,
$r(I)=|\ I\ |= r(K), s.t. \xi_i, \xi_k$ *are equidistributed.*