Infinite Minors characterization of (simple) semimatroids from F. Matus Formalism.

By Alexander Erick Trofimoff

Graduate Research Assistant

EE PhD Candidate Drexel U.

ECE dept Fall 2016

DREXEL UNIVERSITY
Electrical and Computer Engineering
College of Engineering

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Simple semimatroids and semigraphoids

Minor-closed subclasses of semigraphoids originate in a natural way from polymatroids. We examine their forbidden and compulsory minors in this section and discuss the corresponding closure operators and their properties.

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Polymatroid and Matroid functions

A real function h defined on the subsets of N is said to be (the rank function of) a *polymatroid* if
$h(\emptyset) = 0$ and the differences
$\Delta h(i, j|K) = h(iK) + h(jK) - h(ijK) - h(K)$
are nonnegative for all $(i, j|K) \in \mathfrak{S}(N)$,
where the symbol $\mathfrak{S}(N)$ denotes the union of $\mathcal{R}(N)$ and
$\{(i, i|K);\ K \subset N$ and $i \in N - K\}$.
If h is in addition integer-valued and
$h(i) \leq 1$, $i \in N$, then h is a *matroid*,

Matroid functions

The functions $r^L{}_m(K) = \min\{m, |K \cap L|\}, K \subset N$, parametrized by $L \subset N$ and an integer $0 \leq m \leq |L|$ are the most trivial examples of matroids. If $L = N$ the superindex is omitted and the matroids r_m are *uniform*.

SemiMatroids from Polymatroids

Semimatroids are relations $\mathcal{L} \subset \mathcal{S}(N)$ of the form $\mathcal{L}$ = $[h]$
where h is a polymatroid and $[h] = \{(i, j|K) \in \mathcal{S}(N);\ \Delta h(i, j|K) = 0\}$
picks up all equalities in the defining inequalities of polymatroids.

Ex:

$[r^L{}_m] = \{(i, j|K) \in \mathcal{R}(N);\ ij \notin L$ or $|K \cap L| \neq m-1\}$

$\cup\{(i, i|K);\ i \not\subset L$ or $|K \cap L| \geq m\}$

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Matroid as special SemiMatroids

In particular, $[r_0] = [0] = \mathfrak{I}(N)$ and
$[r_{|N|}] = \mathscr{R}(N)$
are semimatroids.
(In fact they are ‘matroids’, since
$h \longrightarrow |[h]|$ is for matroids injective,
Thence, matroids can be taken as special semimatroids.)

Intersection of SemiMatroids

Intersections of semimatroids are semimatroids since the sum h_1 and h_2 is a polymatroid and $[h_1] \cap [h_2] = [h_1 + h_2]$.
To every semimatroid $[h] \subset \mathfrak{S}(N)$
the intersection $[h]$ = $|[h]| \cap \mathscr{R}(N)$ = $|[h+r_{|N|}]|$ is again a semimatroid, which means that we examine mainly traces on $\mathscr{R}(N)$ of semimatroids.
Semimatroids contained in $\mathscr{R}(N)$

Simple and Homogeneous SemiMatroids

A relation $\mathcal{L} \subset \mathcal{R}(N)$ is called
a *simple semimatroid*
if it can be written as the intersection of
semimatroids $[r^{L}{}_{m}]$.
If only semimatroids $[r_m]$ occur in the
intersection we speak about
homogeneous semimatroids.

CI-relations that are Homogeneous SemiMatroids

The class of homogeneous semimatroids is trivial. A relation $\mathcal{L} \subset \mathcal{R}(N)$ is a homogeneous semimatroid

iff

$(i, j|J) \in \mathcal{L}, (k, l|L) \in \mathcal{R}(N)\ and\ |J| = |L| \Longrightarrow (k, l|L) \in \mathcal{L}$

Minors of homogeneous semimatroids are evidently homogeneous.

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Simple and Homogeneous SemiMatroids

A relation $\mathcal{L}$ is homogeneous iff all its 3-minors are homogeneous. In fact, if all 3-minors of $\mathcal{L}$ are homogeneous and $(i, j|L) \in \mathcal{L}$ then

$(i, k|L) \in \mathcal{L}\ \forall k \in N - i|L$

(because the 3-minor $\mathrm{re}_{ijk}\ \mathrm{co}_{N\,-\,L}\mathcal{L}$ is homogeneous)

and $(i, k|j(L - k)) \in \mathcal{L}$

$\forall\ k \in \mathcal{L}$(look at $\mathrm{re}_{ijk}\ \mathrm{co}_{N\,-\,(L\,-\,k)}\mathcal{L}$).

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Lemma

The classes of semimatroids and simple semimatroids are minor-closed.

Proof. If $[h]$ is a semimatroid and $L \subset N$ then $\mathrm{re}_L\,[h] = [g]$
where g is the restriction of the function h
on the power set of L.
Similarly, $\mathrm{co}_L\,[h] = [g]$ where the
function g is given by $g(K) = h(K(N-L)), K \subset L$.

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Lemma

The classes of semimatroids and simple semimatroids are minor-closed.

Proof. (Cont)
In both cases g is a polymatroid and in this way
the restriction and contraction
have been defined for polymatroids.
The
remaining assertions follow from the facts that
minors of intersections are intersections
of minors and that restrictions and contractions of the matroids ${r^L}_m$
are matroids of the same type.

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Consequence 5

The classes of semimatroids and simple semimatroids
cannot be characterized by finite sets of forbidden minors.

we have used for the intersections of the previous Proof only
the simple semimatroids $[r^L{}_1]$ and $[r^L{}_{|L|-1}]$ and
the homogeneous semimatroids $R_0(N)$, $R_1(N)$ and
$R_0(N) \setminus R_1(N)$ = ;
Since $R_0(N)$ is the intersection of $[r^L{}_2]$ $\forall L \subset N$, $|L|$=3
and $R_1(N)$ is the intersection of $[r_1]$ with all $[r^L{}_3]$, $L \subset N$, $|L| = 4$,
every proper minor of the two graphoids of figure 10 can
be written as the intersection of the simple semimatroids
$[r^L{}_1]$ and $[r^L{}_{|L|-1}]$.

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Forbidden 4- minors of semimatroids .

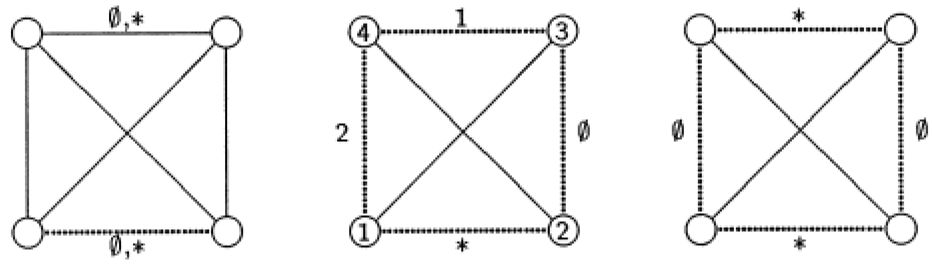

Figure 11. Examples of forbidden 4-minors for semimatroids.

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Consequence 6.

The classes of semimatroids and simple semimatroids cannot be characterized by finite sets of forbidden minors.

Forbidden minors of P-representability

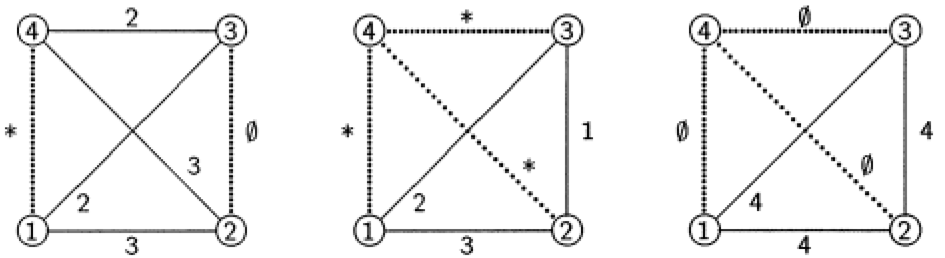

Figure 12. Examples of 4-semimatroids that are not simple.

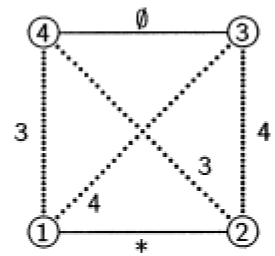

Figure 13. Example of a 4-semimatroid that is not p-representable.

Every 3-semigraphoid is a simple semimatroid.
The forbidden 3-minors of semimatroids are thus the same as the forbidden 3-minors of semigraphoids, (Consequence 2).
By Proposition 1 the forbidden 4-minors of the class of semimatroids are exactly those 4-semigraphoids that are not semimatroids.

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The cone of Polymatroids $H(N)$

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The set of all polymatroids on N is a closed pointed cone $H(N)$ in E^n.
This cone has a finite number of extreme rays.
On each extreme ray a (nonzero and integer-valued) representing polymatroid f can be chosen.
Every polymatroid h on N can be written as $\sum\{\alpha_f f; f \in H_{\text{ext}}(N)\}$,
where f ranges over the set $H_{\text{ext}}(N)$ of the fixed representing polymatroids and α_f are nonnegative numbers.

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The cone of Polymatroids $H(N)$

Then the semimatroid $|[h]|([h])$ is the intersection of the semimatroids $|[f]|$ $([f])$ over those $f \in H_{\text{ext}}(N)$ for which α_f is positive;

The knowledge of all extreme rays therefore increases insight into the class of semimatroids.

The extreme rays in general are far from being completely understood.

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The cone of Polymatroids $H(N)$

The extreme rays of the cone of 4-polymatroids are known

(F. Matus, Extreme convex set functions with many nonnegative differences, Discrete Mathematics V. 135 (1994) 177–191) , in a disguise of the game theory in [L.S. Shapley, Cores of convex games, Int. J. of Game Theory (1971/72) 11–26. and M. Studeny, Convex set functions I. and II., Research Reports 1733 and 1734, Institute of Information Theory and Automation, Prague (November 1991]).

They give rise to the semimatroids [$r^L{}_m$] and to the semimatroids of figures 12 and 13 that are not simple semimatroids.

The cone of Polymatroids $H(N)$

Relying on [F. Matus and M. Studeny, Conditional independences among four random variables I, Combinatorics, Probability and Computing 4 (1995) 269–278.]

we can formulate without proof the following characterization of compulsory 4-minors of the class of semimatroids contained in $\mathcal{R}(N)$.

Proposition 5.

A 4-relation $\mathcal{L} \subset \mathcal{R}(N)$ is a semimatroid iff it can be expressed as the intersection of a simple semimatroid with isomorphic copies of the semimatroids from figures 12 and 13

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Semimatroids Closure operator

Given a relation $\mathcal{L} \subset \mathfrak{S}(N)$ ∃ the smallest semimatroid on N containing $\mathcal{L}$. This one, being the intersection of all semimatroids on N containing $\mathcal{L}$, is denoted by $m(\mathcal{L})$.
if $\mathcal{L} \subset \mathscr{R}$(N) then m($\mathcal{L}$) ⊂ $\mathscr{R}$(N).
The mapping m on the power set of $\mathfrak{S}(N)$ ($\mathscr{R}(N)$) is the closure operator of semimatroids (under $\mathscr{R}(N)$).
A relation $\mathcal{L}$ is a semimatroid iff $\mathcal{L}$ = $m(\mathcal{L})$.

Graphoid Closure operator

The closure operator $\mathfrak{g}$ of the class of graphoids has the following plausible property. $\exists$ a $m \in \mathbb{Z}^{+}$ s.t. $\forall \mathcal{I} \subset \mathcal{R}(N)$

$$\mathcal{I} = \mathfrak{g}(\mathcal{I}) \Leftrightarrow \mathcal{I} = \{\mathfrak{g}(K);\ K \subset \mathcal{I},\ |K| \leq m\}\ .$$

Consequence 7.

Let $\mathbb{c}$ be the closure operator of an intersection-closed class of semimatroids. If $\mathbb{c}$ is between f and m, i.e., $m(\mathcal{L}) \subset \mathbb{c}(\mathcal{L}) \subset f(\mathcal{L})$ for all $\mathcal{L} \subset \mathcal{R}(N)$ (and finite N), then $\mathbb{c}$ is not finitary

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Linear and probabilistic semimatroids.

In this section we investigate special semimatroids related to the lattice of linear subspaces of a linear space.
They provide classes of CI-relations which might be fundamental for a systematic approach to CI-relations.
In addition, they provide a convenient and comfortable tool for a basic study of the notion of p-representability to be done in the second part of this section.

P-representable semimatroids .

The p-representable semimatroids originate from systems of r.v. $\xi = (\xi_i)_{i \in N}$: a relation $\mathcal{M}$ is p-representable if $\exists$ ξ s.t. a triple (I, J, K) belongs to $\mathcal{M}$ iff the subsystems $\xi_I = (\xi_i)_{i \in I}$ and ξ_J are stochastically conditionally independent given the subsystem K.

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l- representable CI-relations.

A relation $\mathcal{M} \in \mathfrak{S}(N)$ is l-representable over the field F if ∃ a collection of subspaces
$D = (Di)_{i \in N}$ of a linear space over F
s.t. a triple (I, J, K) belongs to $\mathcal{M}$
iff $D_{I \cup K} \cup D_{J \cup K} = D_K$ where
$D_K = \bigoplus_{k \in K} D_k$

$\mathcal{M} = |[D]|_{\text{lin}}$
$|[D]|_{\text{lin}} = \{(i, j|K) \in \mathfrak{S}(N);\ D_{IK} \cup D_{JK} = D_K\}$

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Every l- representable is a Semimatroid.

If we denote by $h_D(K)$ the dimension of the linear space $D_K, K \subset N$,
the set function h_D is a polymatroid because
$\Delta\, h_D(i,j|K) = h_D(iK) + h_D(jK) - h_D(ijK) - h_D(K)$
$= \dim(D_{iK} \cap DjK) - \dim(D_K) \geq 0$ and $|[h_D]| = |[D]|_{\text{lin}}$
Every l-representable relation $\mathcal{L} \subset \mathfrak{S}(N)$ is thus a semimatroid.

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